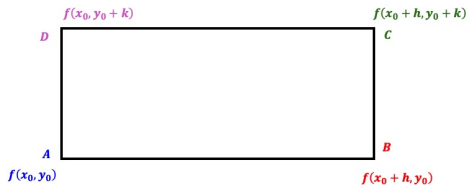


# MATH 224: Vector Calculus



Class 10: March 2, 2026



- ▶ Assignment 10 (For Friday)
- ▶ Limits and Continuity

# Announcements

Exam 1: Tonight, 7 PM -  
No Time Limit

No Books, Notes, Computers, etc.

Warner 104: A - M

Warner 105: N - Z

Today: Complete Proof of Clairault's. Theorem

Today: Begin Chapter 4

Topic: Differentiability

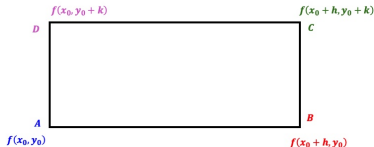
Start with  $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$

Eventually:  $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$

Derivative at point turns out to be  $m \times n$  matrix.

But First: Limits and Continuity

Clairault's Theorem: If the functions  $f, f_x, f_y, f_{xy}$  and  $f_{yx}$  are continuous on  $\mathbb{R}^2$ , then  $f_{xy}$  and  $f_{yx}$  are identical functions



$$F(h, k) = [f(C) - f(B)] - [f(D) - f(A)]$$

$$F(h, k) = [f(x_0 + h, y_0 + k) - f(x_0 + h, y_0)] - [f(x_0, y_0 + k) - f(x_0, y_0)]$$

$$\text{Define } G(x) = f(x, y_0 + k) - f(x, y_0)$$

Apply MVT to  $G$  on the interval  $[x_0, x_0 + h]$ . Result is an  $x_c$  inside the interval such  $hG'(x_c) = G(x_0 + h) - G(x_0)$

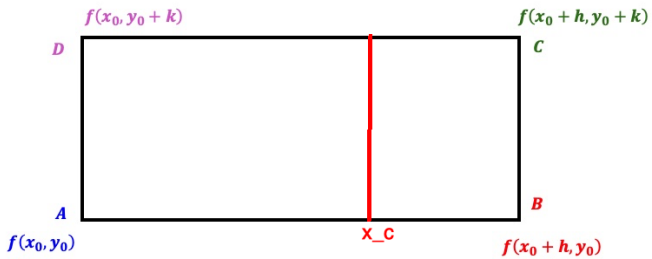
Examine RHS:

$$G(x_0 + h) = f(x_0 + h, y_0 + k) - f(x_0 + h, y_0) = f(C) - f(B)$$

$$\text{and } G(x_0) = f(x_0, y_0 + k) - f(x_0, y_0) = f(D) - f(A)$$

$$\text{so } G(x_0 + h) - G(x_0) = F(h, k).$$

$$\text{LHS: } h G'(x_c) = h[f_x(x_c, y_0 + k) - f_x(x_c, y_0)]$$



Summarizing,

$$hG'(x_c) = G(x_0 + h) - G(x_0) \text{ becomes}$$
$$h[f_x(x_c, y_0 + k) - f_x(x_c, y_0)] = F(h, k)$$

Now examine the function  $H(y) = f_x(x_c, y)$  on the  $y$  interval  
 $[y_0, y_0 + k]$

Applying the MVT yields a number  $y_d$  inside that interval such that

$$k H'(y_d) = H(y_0 + k) - H(y_0)$$
$$k f_{xy}(x_c, y_d) = f_x(x_c, y_0 + k) - f_x(x_c, y_0) = \frac{F(h, k)}{h}$$

so

$$F(h, k) = kh f_{xy}(x_c, y_d)$$

for some point  $(x_c, y_d)$  inside rectangle .

For the second half, begin with noting

$$F(h, k) = [f(C) - f(B)] - [f(D) - f(A)] = [f(C) - f(D)] - [f(A) - f(B)]$$

and following then procedure of the first part, but differentiating first with respect to  $y$  and second with respect to  $x$ , using MVT twice.

Obtain  $F(h, k) = hk f_{yx}(x_{c^*}, y_{d^*})$   
for some point  $(x_{c^*}, y_{d^*})$  inside rectangle.

Thus  $f_{xy}(x_c, y_d) = f_{yx}(x_{c^*}, y_{d^*})$   
By Continuity of  $f_{xy}$  and  $f_{yx}$ , both terms have limit  $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$  as  $(h, k)$  approaches  $(0,0)$

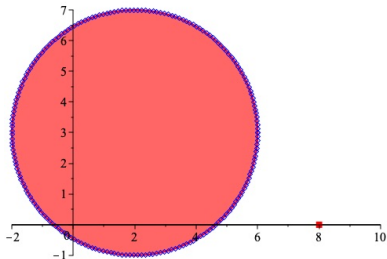
## Limits and Continuity: Preliminary Concepts

Open Set      Interior Point

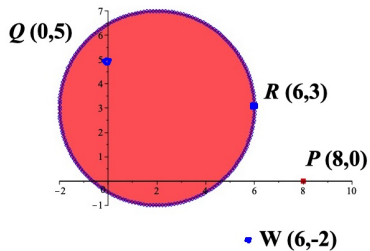
Closed Set    Boundary Point

Limit Point   Neighborhood

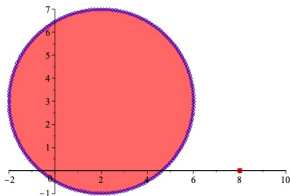
Example:  $S = \{|x - (2, 3)| < 4\} \cup \{(8, 0)\}$



Example:  $S = \{|x - (2, 3)| < 4\} \cup \{(8, 0)\}$



Example:  $S = \{|x - (2, 3)| < 4\} \cup \{(8, 0)\}$



Point	Interior Point?	Limit Point	Boundary Point
Q	Yes	Yes	Yes
R	No	Yes	Yes
P	No	No	Yes
W	No	No	No

Differentiability = Local Linearity = Approximatable By Tangent Object

$$f(x) \approx f(a) + f'(a)(x - a)$$

$$\text{or } f(x) - f(a) \approx f'(a)(x - a)$$

$$\text{or } f(x) - f(a) - m(x - a) \approx 0$$

$$\lim_{x \rightarrow a} \frac{f(x) - f(a) - m(x - a)}{|x - a|} = 0$$

Generalizing for  $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} \frac{\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - M(\mathbf{x} - \mathbf{a})}{|\mathbf{x} - \mathbf{a}|} = \mathbf{0}$$

for some  $m \times n$  matrix  $M$ .

$\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is **differentiable** at  $\mathbf{a}$  if there exists an  $m \times n$  matrix  $M$  such that

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} \frac{\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - M(\mathbf{x} - \mathbf{a})}{|\mathbf{x} - \mathbf{a}|} = \mathbf{0}$$

Special Case:  $m = 1, n = 2, M$  is  $1 \times 2$  matrix  $\nabla f = (f_x, f_y)$ .

Example:  $f(x, y) = x^2 + 2xy - y^2$  at  $(-1, 2)$

$$f(-1, 2) = -7$$

$$f_x(x, y) = 2x + 2y \text{ so } f_x(-1, 2) = 2$$

$$f_y(x, y) = 2x - 2y \text{ so } f_y(-1, 2) = -6$$

$$\nabla f(-1, 2) = (2, -6)$$

Equation of Tangent Plane:

$$z = -7 + (2, -6) \cdot (x + 1, y - 2)$$

$$= -7 + 2x + 2 - 6y + 12$$

$$= +7 + 2x - 6y$$

Review meaning of  $f_x(-1, 2) = 2$  and  $f_y(-1, 2) = 6$

What is rate of change of  $f$  at  $(-1, 2)$  if we approach along direction given by  $\mathbf{v} = (3, 4)$ ?

$$\begin{aligned}f_{\mathbf{v}}(-1, 2) &= \lim_{t \rightarrow 0} \frac{f(-1 + 3t, 2 + 4t) - f(-1, 2)}{t} \\&= \lim_{t \rightarrow 0} \frac{(-1 + 3t)^2 + 2(-1 + 3t)(2 + 4t) - (2 + 4t)^2 - (-7)}{t} \\&= \lim_{t \rightarrow 0} \frac{17t^2 - 18t}{t} \\&= \lim_{t \rightarrow 0} (17t - 18) = -18\end{aligned}$$

Note:  $(\nabla f) \cdot \mathbf{v} = (2, -6) \cdot (3, 4) = (2)(3) + (-6)(4) = 6 - 24 = -18$

**COINCIDENCE?**