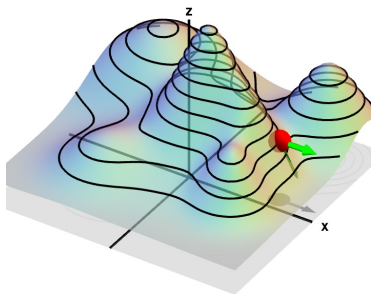



# MATH 224: Vector Calculus



$\theta = 0$   
  
 $u = (1.00, 0.00)$

$D_u f(a) = -1.81$   
  
 $a = (6.7, 1.1)$

Class 11: March 4, 2026



Notes on Exam 1  
Notes on Assignment 9  
Assignment 10

Exam 1 Results  
Median Grade: 90.5

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}$$

means

For every  $\epsilon$ -neighborhood  $V$  of  $\mathbf{b}$ , there is an  $\delta$ -neighborhood  $U$  of  $\mathbf{a}$  such that  $\mathbf{x}$  in  $U$  ( $\mathbf{x} \neq \mathbf{a}$ ) implies  $\mathbf{f}(\mathbf{x})$  is in  $V$ .

A function  $\mathbf{f}$  is **continuous** at  $\mathbf{a}$  if there is a  $\mathbf{b}$ , such that

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}$$

and

$$\mathbf{f}(\mathbf{a}) = \mathbf{b}$$

*Theorem:* Limits are Unique:

If  $\lim_{\mathbf{x} \rightarrow \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}$  and  $\lim_{\mathbf{x} \rightarrow \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{c}$ , then  $\mathbf{b} = \mathbf{c}$

*Theorem:* Given  $f : R^n \rightarrow R^m$  with coordinate functions  $f_1, f_2, \dots, f_m$  and a point  $\mathbf{y}_1 = (y_1, y_2, \dots, y_m)$  in  $R^m$ , then  $\lim_{\mathbf{x} \rightarrow \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{y}_1$  if and only if  $\lim_{\mathbf{x} \rightarrow \mathbf{a}} f_i(\mathbf{x}) = y_i$  for  $i = 1, 2, \dots, m$ .

## Algebra of Limits

*If  $\lim_{x \rightarrow a} f(x)$  and  $\lim_{x \rightarrow a} g(x)$  exists*

Today: Begin Chapter 4

Topic: Differentiability

Start with  $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$

Eventually:  $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$

Derivative at point turns out to be  $m \times n$  matrix.

But First: Limits and Continuity

Differentiability = Local Linearity = Approximatable By Tangent Object

$$f(x) \approx f(a) + f'(a)(x - a)$$

$$\text{or } f(x) - f(a) \approx f'(a)(x - a)$$

$$\text{or } f(x) - f(a) - m(x - a) \approx 0$$

$$\lim_{x \rightarrow a} \frac{f(x) - f(a) - m(x - a)}{|x - a|} = 0$$

Generalizing for  $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} \frac{\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - M(\mathbf{x} - \mathbf{a})}{|\mathbf{x} - \mathbf{a}|} = \mathbf{0}$$

for some  $m \times n$  matrix  $M$ .

$\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is **differentiable** at  $\mathbf{a}$  if there exists an  $m \times n$  matrix  $M$  such that

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} \frac{\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - M(\mathbf{x} - \mathbf{a})}{|\mathbf{x} - \mathbf{a}|} = \mathbf{0}$$

Special Case:  $m = 1, n = 2, M$  is  $1 \times 2$  matrix  $\nabla f = (f_x, f_y)$ .

Example:  $f(x, y) = x^2 + 2xy - y^2$  at  $(-1, 2)$

$$f(-1, 2) = -7$$

$$f_x(x, y) = 2x + 2y \text{ so } f_x(-1, 2) = 2$$

$$f_y(x, y) = 2x - 2y \text{ so } f_y(-1, 2) = -6$$

$$\nabla f(-1, 2) = (2, -6)$$

Equation of Tangent Plane:

$$z = -7 + (2, -6) \cdot (x + 1, y - 2)$$

$$= -7 + 2x + 2 - 6y + 12$$

$$= +7 + 2x - 6y$$

Review meaning of  $f_x(-1, 2) = 2$  and  $f_y(-1, 2) = 6$

What is rate of change of  $f$  at  $(-1, 2)$  if we approach along direction given by  $\mathbf{v} = (3, 4)$ ?

$$\begin{aligned}f_{\mathbf{v}}(-1, 2) &= \lim_{t \rightarrow 0} \frac{f(-1 + 3t, 2 + 4t) - f(-1, 2)}{t} \\&= \lim_{t \rightarrow 0} \frac{(-1 + 3t)^2 + 2(-1 + 3t)(2 + 4t) - (2 + 4t)^2 - (-7)}{t} \\&= \lim_{t \rightarrow 0} \frac{17t^2 - 18t}{t} \\&= \lim_{t \rightarrow 0} (17t - 18) = -18\end{aligned}$$

Note:  $(\nabla f) \cdot \mathbf{v} = (2, -6) \cdot (3, 4) = (2)(3) + (-6)(4) = 6 - 24 = -18$

**COINCIDENCE?**

## Major Theorems

If  $\mathbf{f}$  is differentiable at  $\mathbf{a}$ , then  $\mathbf{f}$  is continuous at  $\mathbf{a}$ .

If all partial derivatives of  $\mathbf{f}$  are continuous in a neighborhood of  $\mathbf{a}$ , then  $\mathbf{f}$  is differentiable at  $\mathbf{a}$ .

If  $\mathbf{f}$  is differentiable at  $\mathbf{a}$ , then  $M$  is the matrix of first order partial derivatives.

## Partial With Respect to a Vector

Let  $f(x, y) = x^2y$  and  $\mathbf{a} = (3, 9)$  so  $f(3, 9) = 81$ .

Find the partial derivative of  $f$  at  $(3, 9)$  if we approach  $(3, 9)$  along arbitrary vector  $\mathbf{v} = (v_1, v_2)$ .

$$\text{We want } f_{\mathbf{v}}(\mathbf{a}) = \lim_{t \rightarrow 0} \frac{f(\mathbf{a} + t\mathbf{v}) - f(\mathbf{a})}{t}$$

$$\begin{aligned} f_{\mathbf{v}}(\mathbf{a}) &= \lim_{t \rightarrow 0} \frac{f(3 + tv_1, 9 + tv_2) - f(3, 9)}{t} \\ &= \lim_{t \rightarrow 0} \frac{(3 + tv_1)^2(9 + tv_2) - (3^2)(9)}{t} \\ &= \lim_{t \rightarrow 0} \frac{(3^2 + 6tv_1 + t^2v_1^2)(9 + tv_2) - (3^2 \cdot 9)}{t} \\ &= \lim_{t \rightarrow 0} \frac{(3^2)(9) + 3^2tv_2 + 6tv_1(9) + 6t^2v_1v_2 + t^2v_1^2(9) + t^3v_1^2v_2 - (3^2 \cdot 9)}{t} \\ &= \lim_{t \rightarrow 0} (3^2v_2 + 54v_1 + t6v_1v_2 + tv_1^2(9) + t^2v_1^2v_2) \\ &= 9v_2 + 54v_1 = 54v_1 + 9v_2 = (54, 9) \cdot (v_1, v_2) \end{aligned}$$