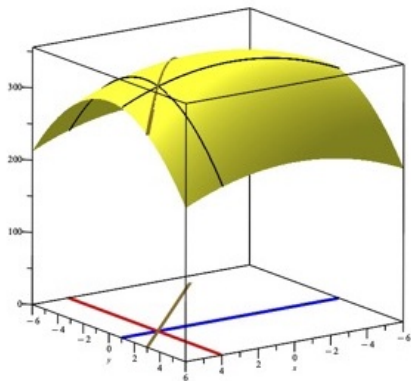


MATH 224: Vector Calculus



Class 13: Monday, March 9, 2026



- ▶ Notes on Assignment 11
- ▶ Assignment 12



- ▶ Drop In Tutoring with Zane Malarkey
Tuesday 7 – 9 PM
Second Floor of Warner
- ▶ Friday's and Next Monday's Class on Zoom

Definition: $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$ is **differentiable at \mathbf{a}** point in \mathbb{R}^n if there is a $1 \times n$ matrix M such that

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} \frac{f(\mathbf{x}) - f(\mathbf{a}) - M(\mathbf{x} - \mathbf{a})}{|\mathbf{x} - \mathbf{a}|} = 0$$

$$\lim_{|\mathbf{h}| \rightarrow 0} \frac{f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - M\mathbf{h}}{|\mathbf{h}|} = 0$$

The vector M is called the **gradient** of f at \mathbf{a} , denoted at $\nabla f(\mathbf{a})$.

Theorem If f is differentiable at \mathbf{a} , then

$$\nabla f(\mathbf{a}) = [f_{x_1}(\mathbf{a}), f_{x_2}(\mathbf{a}), \dots, f_{x_n}(\mathbf{a})]$$

Directional Derivative

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^1$$

\mathbf{a} point and \mathbf{v} vector in \mathbb{R}^n

Find the directional derivative of f at \mathbf{a} in the direction of the vector \mathbf{v} is

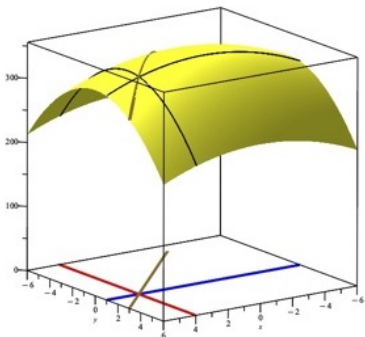
$$f_{\mathbf{u}}(\mathbf{a}) \text{ where } \mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

Rate of Change in Direction \mathbf{u} is

$$\nabla f(\mathbf{a}) \cdot \mathbf{u} = |\nabla f(\mathbf{a})| |\mathbf{u}| \cos \theta = |\nabla f(\mathbf{a})| \cos \theta$$

since $|\mathbf{u}| = 1$. Maximum rate of change occurs when $\cos \theta = 1$; that is $\theta = 0$ so pick \mathbf{u} in the direction of the gradient.

Maximum Rate of Change Example



$$f(x, y) = 356 - x^2 - 3y^2 \text{ at } \mathbf{a} = (4, 1)$$

$$f(\mathbf{a}) = f(4, 1) = 337$$

$$f_x(x, y) = (-2x), f_y(x, y) = -6y \text{ so } f_x(\mathbf{a}) = -8, f_y(\mathbf{a}) = -6$$

$$\nabla f(\mathbf{a}) = (-8, -6)$$

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\mathbf{v}	\mathbf{u}	Rate of Change
$(-1, 0)$	$(-1, 0)$	8
$(0, -1)$	$(0, -1)$	6
$(-1, -1)$	$(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$	$7\sqrt{2} = 9.89949$
$(-3, -2)$	$(\frac{-3}{\sqrt{13}}, \frac{-2}{\sqrt{13}})$	$\frac{36}{\sqrt{13}} = 9.89446$
$(-8, -6)$	$(\frac{-8}{\sqrt{100}}, \frac{-6}{\sqrt{100}})$	10

Some Big Question About Differentiability

- ▶ What rules of differentiability for $f : \mathbb{R}^1 \rightarrow \mathbb{R}^1$ carry over for $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$?
Sums, Differences, Constant Multiples, Products, Quotients?
- ▶ What is the analogue of the Mean Value Theorem?
- ▶ Is There a Chain Rule for Derivative of a Composition?
- ▶ How Can We Define Differentiability for $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$?

Generalized Mean Value Theorem for $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$

If f is differentiable at each point of a line segment S between \mathbf{a} and \mathbf{b} , then there is at least one point \mathbf{c} on S such that

$$f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(\mathbf{c}) \cdot (\mathbf{b} - \mathbf{a})$$

Recall classic MVT from Single Variable Calculus:

If $f : \mathbb{R}^1 \rightarrow \mathbb{R}^1$ is differentiable on a closed interval $[a, b]$, then there is at least one c inside the interval such that

$$f(b) - f(a) = f'(c)(b - a).$$

An Important Consequence of classic MVT:

Suppose $f'(x) = g'(x)$ for all x in $[a, b]$. Then $f(x) = g(x) + C$ for some constant C and all x in the interval.

Proof: Let $H(x) = f(x) - g(x)$.

Then $H'(x) = f'(x) - g'(x) = 0$ for all x in the interval.

Now let $x_1 < x_2$ be any two points in the interval.

By MVT: $H(x_2) - H(x_1) = H'(c)(x_2 - x_1) = 0(x_2 - x_1) = 0$. Thus

H is a constant function: $H(x) = C$ for all x .

So $f(x) - g(x) = C$ and hence $f(x) = g(x) + C$.

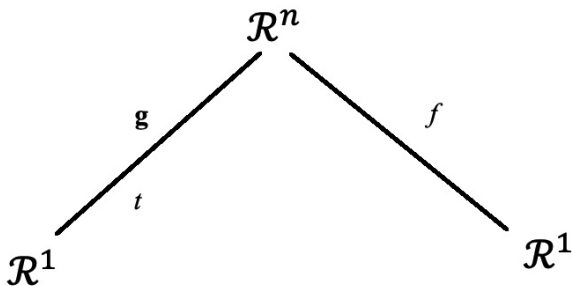
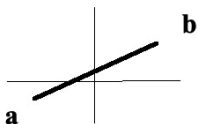
The same argument shows

$$\nabla f \equiv \nabla g \text{ implies } f(\mathbf{x}) = g(\mathbf{x}) + C$$

Generalized Mean Value Theorem for $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$

If f is differentiable at each point of a line segment S between \mathbf{a} and \mathbf{b} , then there is a least point \mathbf{c} on S such that

$$f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(\mathbf{c}) \cdot (\mathbf{b} - \mathbf{a})$$



Proof of Generalized Mean Value Theorem

Define a new function $\mathbf{g} : [0, 1] \rightarrow \mathbb{R}^n$ by $\mathbf{g}(t) = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$
Note $\mathbf{g}(0) = \mathbf{a}$ and $\mathbf{g}(1) = \mathbf{b}$ and $\mathbf{g}(t)$ lies on S and $\mathbf{g}'(t) = \mathbf{b} - \mathbf{a}$

Consider the composition $H(t) = f(\mathbf{g}(t)) : [0, 1] \rightarrow \mathbb{R}^1$

Apply Classic MVT to H :

$$H(1) - H(0) = H'(t_c)(1 - 0) = H'(t_c)$$

but $H(1) = f(\mathbf{g}(1)) = f(\mathbf{b})$ and $H(0) = f(\mathbf{g}(0)) = f(\mathbf{a})$

Thus $f(\mathbf{b}) - f(\mathbf{a}) = H'(t_c)$

What is $H'(t)$? By Chain Rule: $f'(\mathbf{g}(t))\mathbf{g}'(t) = \nabla f(\mathbf{g}(t)) \cdot (\mathbf{b} - \mathbf{a})$

Let $\mathbf{C} = \mathbf{g}(t_c)$. Then

$$f(\mathbf{b}) - f(\mathbf{a}) = H'(t_c) = \nabla f(\mathbf{C}) \cdot (\mathbf{b} - \mathbf{a})$$