MATH 224: Vector Calculus

Class 14: Wednesday, March 11, 2026

- Notes on Assignment 12
- Assignment 13

Friday and Monday Classes on Zoom

Today

Differentiability of Vector - Valued Functions of Vectors

Chain Rule

Implicit Differentiation

Differentiable Function

Definition: $f : \mathbb{R}^n \to \mathbb{R}^1$ is **differentiable at a point** $\mathbf{a}$ in $\mathbb{R}^n$ if there is a **1 by n matrix M** such that

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{f(\mathbf{x}) - f(\mathbf{a}) - M(\mathbf{x}-\mathbf{a})}{|\mathbf{x}-\mathbf{a}|} = 0$$

The $1 \times n$ matrix M is called the **gradient** or **derivative** of f at $\mathbf{a}$, denoted as $f'(\mathbf{a}) = \nabla f(\mathbf{a})$.

Theorem If f is differentiable at $\mathbf{a}$, then

$$\nabla f(\mathbf{a}) = \left[f_{x_1}(\mathbf{a}), f_{x_2}(\mathbf{a}), \ldots, f_{x_n}(\mathbf{a})\right]$$

Definition $f : \mathbb{R}^n \to \mathbb{R}^m$ is **differentiable at a point** point $\mathbf{a}$ in $\mathbb{R}^n$ if there is a **m by n matrix M** such that

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{f(\mathbf{x}) - f(\mathbf{a}) - M(\mathbf{x}-\mathbf{a})}{|\mathbf{x}-\mathbf{a}|} = 0$$

How Shall We Compute This m by n matrix M?

Recall if $\mathbf{F} : \mathbb{R}^1 \to \mathbb{R}^m$ so that $\mathbf{F}(t) = (f_1(t), f_2(t), \ldots, f_m(t))$, then $\mathbf{F}'(t) = (f_1'(t), f_2'(t), \ldots, f_m'(t))$,

Example: Suppose $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^3$ is a differentiable function. Then $\mathbf{F}(x, y) = (f(x, y), g(x, y), h(x, y))$ where each of f, g, h is a differentiable function of x and y. More properly, we should write $\mathbf{F}$ as

$$\mathbf{F}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} f(x, y) \\ g(x, y) \\ h(x, y) \end{bmatrix}$$

Since f, g, h can be completely independent of each other, we can form the derivative of $\mathbf{F}$ by independently taking the derivative of each of the component functions

Example: Suppose $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^3$ is a differentiable function:

$$\mathbf{F}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} f(x,y) \\ g(x,y) \\ h(x,y) \end{bmatrix}$$

$$\mathbf{F}'\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \nabla f(x,y) \\ \nabla g(x,y) \\ \nabla h(x,y) \end{bmatrix} = \begin{bmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \\ h_x(x,y) & h_y(x,y) \end{bmatrix}$$

General Case

$\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$ has m component functions $f_1, f_2, \ldots, f_m$ each of which is a real-valued function of n variables.

$\mathbf{F}$ is differentiable if and only each of its component functions is differentiable, and

$$\mathbf{F}' = \begin{bmatrix} f_{1_{x_1}} & f_{1_{x_2}} & \ldots & f_{1_{x_j}} & \ldots & f_{1_{x_n}} \\ f_{2_{x_1}} & f_{2_{x_2}} & \ldots & f_{2_{x_j}} & \ldots & f_{2_{x_n}} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ f_{i_{x_1}} & f_{i_{x_2}} & \ldots & f_{i_{x_j}} & \ldots & f_{i_{x_n}} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ f_{m_{x_1}} & f_{m_{x_2}} & \ldots & f_{m_{x_j}} & \ldots & f_{m_{x_n}} \end{bmatrix}$$

$\mathbf{F}'$ is an $m \times n$ matrix whose ijth entry is the partial of the ith component functions with respect to the jth variable.

This derivative matrix $\mathbf{F}' = \begin{bmatrix} f_{1_{x_1}} & f_{1_{x_2}} & \dots & f_{1_{x_j}} & \dots & f_{1_{x_n}} \\ f_{2_{x_1}} & f_{2_{x_2}} & \dots & f_{2_{x_j}} & \dots & f_{2_{x_n}} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ f_{i_{x_1}} & f_{i_{x_2}} & \dots & f_{i_{x_j}} & \dots & f_{i_{x_n}} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ f_{m_{x_1}} & f_{m_{x_2}} & \dots & f_{m_{x_j}} & \dots & f_{m_{x_n}} \end{bmatrix}$

is called the **Jacobian Matrix**

If $m = n$, then the determinant of $\mathbf{F}'$ is called the **Jacobian**.

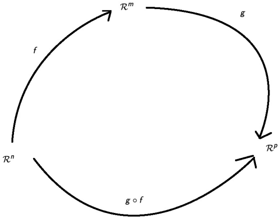
Theorem 4.5.1: If $\mathbf{F}$ is differentiable at $\mathbf{a}$, then i, jth entry of M is the j partial derivative of the ith component of $\mathbf{F}$ evaluated at $\mathbf{a}$

Theorem 4.5.2: If each partial derivative f_{i_j} is continuous, then $\mathbf{F}$ is differentiable.

Carl Gustav Jacob Jacobi
Born: December 10, 1804
Died: February 18, 1851
Jacobi Biography

James Joseph Sylvester
Born: September 3, 1814
Died: March 15, 1897
Sylvester Biography

The Chain Rule



$$(g \circ f)' = g'(f(x))f'(x)$$

(p x m) (m x n)

matrix matrix

p x n matrix

Example Find $(g \circ f)'$ at $(2,3) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ if

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + xy + 1 \\ y^2 + 2 \end{pmatrix}, g\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u + v \\ 2u \\ v^2 \end{pmatrix}$$

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Step I: $f\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2^2 + 6 + 1 \\ 9 + 2 \end{pmatrix} = \begin{pmatrix} 11 \\ 11 \end{pmatrix}$

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$$g'\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 2v \end{pmatrix} \text{ so } g'\begin{pmatrix} 11 \\ 11 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 22 \end{pmatrix}$$

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$$f'\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y & x \\ 0 & 2y \end{pmatrix} \text{ so } f'\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ 0 & 6 \end{pmatrix}$$

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Here we can actually check by direct computation:

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$$g(f(x, y)) = g \begin{pmatrix} x^2 + xy + 1 \\ y^2 + 2 \end{pmatrix} = \begin{pmatrix} x^2 + xy + 1 + y^2 + 2 \\ 2x^2 + 2xy + 2 \\ y^4 + 4y^2 + 4 \end{pmatrix}$$

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$$(g \circ f)' \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y & x + 2y \\ 4x + 2y & 2x \\ 0 & 4y^3 + 8y \end{pmatrix}$$

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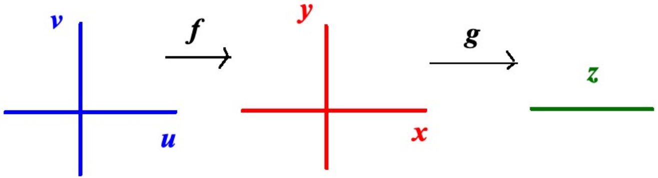
$$(g \circ f)' \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 + 3 & 2 + 6 \\ 8 + 6 & 4 \\ 0 & 108 + 24 \end{pmatrix} = \begin{pmatrix} 7 & 2 + 6 \\ 14 & 4 \\ 0 & 132 \end{pmatrix}$$

Another Example: Suppose $x = u^2 - v^2, y = 2uv$ and $z = g(x, y)$ for some real-valued differentiable function g.

Show $(z_u)^2 + (z_v)^2 = 4(u^2 + v^2)[(z_x)^2 + (z_y)^2]$



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$$\text{Show } (z_u)^2 + (z_v)^2 = 4(u^2 + v^2)[(z_x)^2 + (z_y)^2]$$

$$\text{Let } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u^2 - v^2 \\ 2uv \end{pmatrix} = f \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\text{Then } f' \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix}, g' \begin{pmatrix} x \\ y \end{pmatrix} = (g_x, g_y) = (z_x, z_y)$$

$$\text{Now } (g \circ f)' = g'(f)f' = (z_x, z_y) \begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix} =$$

$$(2uz_x + 2vz_y, -2vz_x + 2uz_y) = (z_u, z_v)$$

Thus

$$\begin{aligned} z_u^2 + z_v^2 &= 4u^2 z_x^2 + 8uvz_x x_y + 4v^2 z_y^2 + 4v^2 z_x^2 - 8uvz_x z_y + 4u^2 z_y^2 \\ &= 4u^2(z_x^2 + z_y^2) + 4v^2(z_x^2 + z_y^2) = 4(u^2 + v^2)(z_x^2 + z_y^2) \end{aligned}$$

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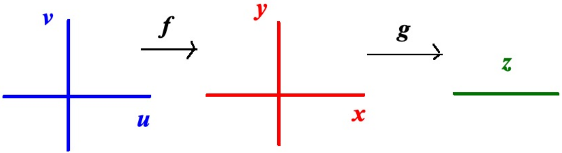


Proving The Chain Rule

Chain Rule for $\mathbf{f} : \mathbb{R}^1 \to \mathbb{R}^n$ and $g := \mathbb{R}^n \to \mathbb{R}^1$

Theorem: Little General Chain Rule Suppose $\mathbf{f} : \mathbb{R}^1 \to \mathbb{R}^n$ is differentiable on an open interval containing a with values in an open set S of $\mathbb{R}^n$. If g is a real-valued function continuously differentiable on S, then the composition $F(x) = (g \circ \mathbf{f})(x)$ is differentiable at a with $F'(a) = \nabla g(\mathbf{f}(a)\mathbf{f}'(a)$.

Chain Rule for $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{g} := \mathbb{R}^n \to \mathbb{R}^p$

Theorem: General Chain Rule Suppose $\mathbf{f}$ is differentiable near $\mathbf{x}$ and $\mathbf{g}$ is continuously differentiable near $\mathbf{f}(\mathbf{x})$. If $\mathbf{g} \circ \mathbf{f}$ is defined on an open set S containing $\mathbf{x}$ then the composition $\mathbf{g} \circ \mathbf{f}$ is continuously differentiable at $\mathbf{x}$ with $(\mathbf{g} \circ \mathbf{f})'(\mathbf{x}) = \mathbf{g}'(\mathbf{f}(\mathbf{x}))\, \mathbf{f}'(\mathbf{x})$.

Implicit Differentiation

Example: Find slope of tangent line to the graph of
$4x^2 + 5y^2 = 61$ at (2,3).
(Check point lies on curve: $4(2^2) + 5(3^2) = 16 + 45 = 61$)

A: Direct Solution

$$5y^2 = 61 - 4x^2 \Rightarrow y^2 = \frac{61 - 4x^2}{5} \Rightarrow y = \sqrt{\frac{61 - 4x^2}{5}}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{61 - 4x^2}{5}\right)^{-1/2} \frac{-8x}{5}$$

Evaluate at $x = 2$: to get $\frac{1}{2}\left(\frac{45}{5}\right)^{-1/2} \frac{-16}{5} = -\frac{8}{15}$

Implicit Differentiation

Example: Find slope of tangent line to the graph of $4x^2 + 5y^2 = 61$ at (2,3).

B: Classic Implicit Differentiation

Treat y as an unknown function of x and differentiate:

$$8x + 10y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-8x}{10y} = -\frac{4x}{5y}$$

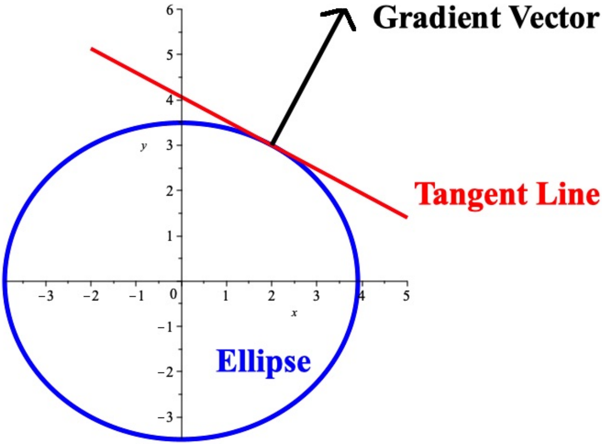
Evaluate at $x = 2, y = 3$: to get $-\frac{8}{15}$

C: Use Level Curve Idea

If $f(x, y) = 4x^2 + 5y^2$, then (2,3) lies on level curve $f(x, y) = 61$. Then $\nabla f(2, 3)$ is normal to the curve so slope of tangent line is the negative of the slope of the gradient.

$\nabla f(x, y) = (8x, 10y)$ has slope $\frac{10y}{8x} = \frac{15}{8}$ at (2,3).

Hence slope of tangent line is $-\frac{8}{15}$.


Gradient Vector
Tangent Line
Ellipse
x
y