

MATH 224: Vector Calculus

INVERSE FUNCTION THEOREM

THEOREM: $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ IS (LOCALLY!) INVERTIBLE NEAR $f(\mathbf{a})$
IF THE DERIVATIVE OF f AT \mathbf{a} IS INVERTIBLE
(AND WE KNOW ALREADY WHAT THE DERIVATIVE OF THE INVERSE IS)

SO, WHEN IS THE DERIVATIVE (& HENCE f) INVERTIBLE?

THAT IS... f IS INVERTIBLE NEAR $f(\mathbf{a})$ IF
 $\text{DET } [Df]_{\mathbf{a}} \neq 0$

THE FINE PRINT:
THE PARTIAL
DERIVATIVES MUST
EXIST AND BE
CONTINUOUS

Class 15: March 13, 2026



Pi Day (3/14) is Tomorrow!



- ▶ Notes on Assignment 13
- ▶ Assignment 14



Monday's Class Also on Zoom
Solutions and Class Notes Are Online

Implicit Differentiation

Change of Variable

Inverse Function Theorem

Gradient Fields

Implicit Differentiation

Example: Find slope of tangent line to the graph of
 $4x^2 + 5y^2 = 61$ at $(2,3)$.

(Check point lies on curve: $4(2^2) + 5(3^2) = 16 + 45 = 61$)

A: Direct Solution

$$5y^2 = 61 - 4x^2 \Rightarrow y^2 = \frac{61 - 4x^2}{5} \Rightarrow y = \sqrt{\frac{61 - 4x^2}{5}}$$

$$\frac{dy}{dx} = \frac{1}{2} \left(\frac{61 - 4x^2}{5} \right)^{-1/2} \frac{-8x}{5}$$

Evaluate at $x = 2$: to get $\frac{1}{2} \left(\frac{45}{5} \right)^{-1/2} \frac{-16}{5} = -\frac{8}{15}$

Implicit Differentiation

Example: Find slope of tangent line to the graph of $4x^2 + 5y^2 = 61$ at $(2,3)$.

B: Classic Implicit Differentiation

Treat y as an unknown function of x and differentiate:

$$8x + 10y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-8x}{10y} = -\frac{4x}{5y}$$

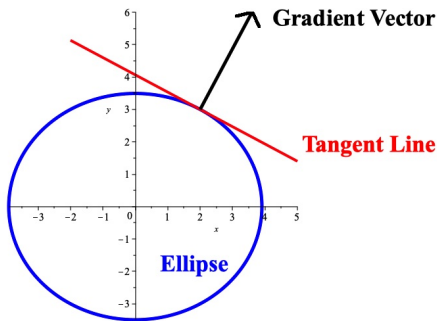
Evaluate at $x = 2, y = 3$: to get $-\frac{8}{15}$

C: Use Level Curve Idea

If $f(x, y) = 4x^2 + 5y^2$, then $(2,3)$ lies on level curve $f(x, y) = 61$. Then $\nabla f(2, 3)$ is normal to the curve so slope of tangent line is the negative of the slope of the gradient.

$\nabla f(x, y) = (8x, 10y)$ has slope $\frac{10y}{8x} = \frac{15}{8}$ at $(2,3)$. Hence slope of tangent line is $-\frac{8}{15}$.

Example: Find slope of tangent line to the graph of $4x^2 + 5y^2 = 61$ at $(2,3)$.



The ellipse is the level curve $F(x, y) = 61$ or $F(x, y) - 61 = 0$ where $F(x, y) = 4x^2 + 5y^2$.

A piece of the curve around $(2, 3)$ is the graph of some implicit function $y = f(x)$.

We want $f'(2)$.

Define a new function $\mathbf{G} : \mathbb{R}^1 \rightarrow \mathbb{R}^2$ by

$$\mathbf{G}(x) = \begin{pmatrix} x \\ f(x) \end{pmatrix} \text{ so } \mathbf{G}'(x) = \begin{pmatrix} 1 \\ f'(x) \end{pmatrix}$$

Note that this is the **tangent vector**.

Then $(F \circ \mathbf{G})(x) = 61$ for all x

Take Derivative Using The Chain Rule:

$$F'(\mathbf{G}(x))\mathbf{G}'(x) = 0. \text{ Thus } \nabla F(\mathbf{G}(x)) \begin{pmatrix} 1 \\ f'(x) \end{pmatrix} = 0$$

Now $G(2) = 3$ and $F(x, y) = 4x^2 + 5y^2$ implies
 $\nabla F(x, y) = (8x, 10y)$.

Hence $\nabla F(G(2)) = (8 \times 2, 10 \times 3) = (16, 30)$.

We have $(16, 30) \begin{pmatrix} 1 \\ f'(2) \end{pmatrix} = 0$ so $16 + 30f'(2) = 0$ and thus
 $f'(2) = -16/30 = -8/15$.

Change of Variable

Example: Find $\int (10x + 15)^{1/3} dx$

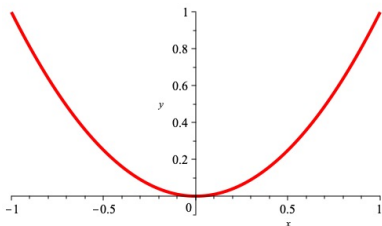
Change of Variable $u = 10x + 15$ so $x = \frac{u-15}{10}$ and $dx = \frac{1}{10} du$

$$\begin{aligned} \text{Integral becomes } \int (10x + 15)^{1/3} dx &= \int u^{1/3} \frac{1}{10} du = \frac{1}{10} \int u^{1/3} du \\ &= \frac{1}{10} \times \frac{3}{4} u^{4/3} + C \\ &= \frac{3}{40} (10x + 15)^{4/3} + C \end{aligned}$$

$x = \frac{u-15}{10}$ is key step. WE MUST BE ABLE TO INVERT THE SUBSTITUTION.

Change of Variable should be invertible, a one-to-one function.

Not Every Function is Invertible

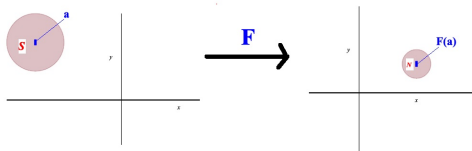


If $y = x^2$, we can not solve unambiguously for x in terms of y globally

$$x = \pm\sqrt{y}$$

but we can solve locally except at origin.

Inverse Function Theorem for $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$



IF

- ▶ \mathbf{a} is a point in \mathbb{R}^n
- ▶ S is an open set containing \mathbf{a}
- ▶ \mathbf{f} is continuously differentiable on S
- ▶ Derivative Matrix $\mathbf{f}'(\mathbf{a})$ is invertible

Then

There is a neighborhood N of \mathbf{a} on which \mathbf{f}^{-1} is defined and

$$(\mathbf{f}^{-1}(\mathbf{f}(\mathbf{x})))' = [\mathbf{f}'(\mathbf{x})]^{-1} \text{ for all } \mathbf{x} \text{ in } N$$

Example: $\mathbf{f}(x, y) = (\cos x, x \cos x - y)$

$$J = \mathbf{f}'(x, y) = \begin{pmatrix} -\sin x & 0 \\ \cos x - x \sin x & -1 \end{pmatrix}$$

$\det J = \sin x$ so we have invertibility if $x \neq 0, \pi$.

$$(\mathbf{f}^{-1}(x, y))' = J^{-1} = \begin{pmatrix} \frac{-1}{\sin x} & 0 \\ \frac{x \sin x - \cos x}{\sin x} & -1 \end{pmatrix}$$

At $x = \pi/6, y = 2$:

$$f\left(\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{\pi}{6} \frac{\sqrt{3}}{2} - 2\right) = \left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}\pi}{12} - 2\right)$$

and

$$\mathbf{f}^{-1}(\pi/6, 2)' = \begin{pmatrix} -2 & 0 \\ \frac{\pi}{6} - \sqrt{3} & -1 \end{pmatrix}$$

Gradient Fields

A **Gradient Field** is just a function from \mathbb{R}^n to \mathbb{R}^n which is the gradient of a differentiable real-valued function.

The gradient $\nabla f(x, y)$ of $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$.

Example 1: $f(x, y) = x^2 \sin y$

Here $\nabla f(x, y) = (2x \sin y, x^2 \cos y) = (f_x(x, y), f_y(x, y))$

Note $f_{xy} = 2x \cos y = f_{yx}$ [**Equality of Mixed Partial**s]

Example 2: Is $\mathbf{F}(x, y) = (y, 2x)$ a gradient field?

If $\mathbf{F} = \nabla f$, then

$$f_x(x, y) = y \implies f_{xy}(x, y) = 1$$

$$f_y(x, y) = 2x \implies f_{yx}(x, y) = 2$$

But these are not equal!

What if we try to build an f by "**Partial Integration**"?

$$f_x(x, y) = y \implies f(x, y) = xy + G(y) \implies f_y(x, y) = x + G'(y)$$

but we would need G a function of y such that $G'(y) = x$.

We can work backwards on Example 1:

Suppose we know

$\nabla f(x, y) = (2x \sin y, x^2 \cos y) = (f_x(x, y), f_y(x, y))$ but we don't
know $f(x, y)$

Given $f_x(x, y) = 2x \sin y$, "partial integration" with respect to x
produces $f(x, y) = x^2 \sin y + G(y)$
and that yields $f_y = x^2 \cos y + G'(y)$
which equals $x^2 \cos y$
by choosing G to be any constant function.