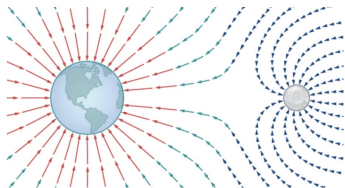
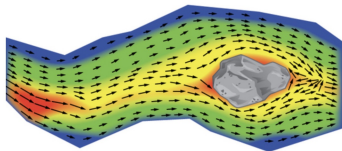


MATH 224: Vector Calculus



(a)



(b)

Class 16: Monday, March 16, 2026



- ▶ Notes on Assignment 14
- ▶ Assignment 15

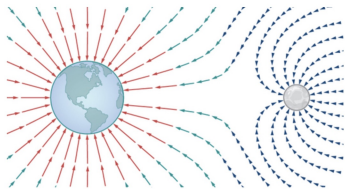
Vector Fields and Gradient Fields

Implicit Differentiation II

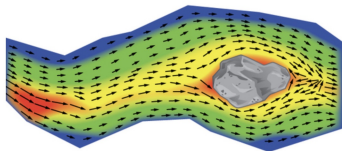
Implicit Function Theorem

Vector Fields

A **Vector Field** is just a function \mathbf{F} from \mathbb{R}^n to \mathbb{R}^n



(a)



(b)

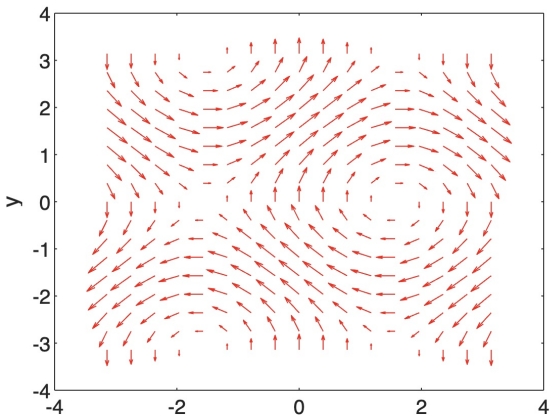
(a) The gravitational field exerted by two astronomical bodies on a small object. (b) The vector velocity field of water on the surface of a river shows the varied speeds of water. Red indicates that the magnitude of the vector is greater, so the water flows more quickly; blue indicates a lesser magnitude and a slower speed of water flow.

Example: $\mathbf{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$\mathbf{F}(x, y) = (\sin y, \cos x)$$

In MATLAB

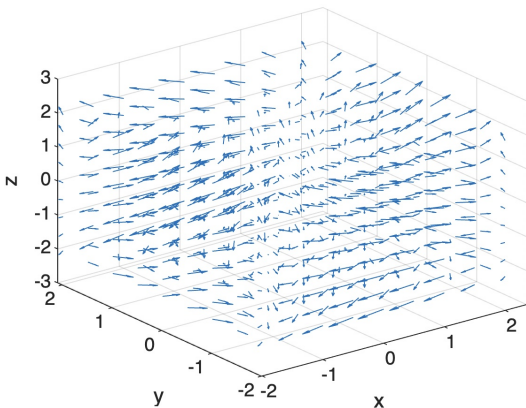
```
[x,y] = meshgrid(-pi:pi/8:pi,-pi:pi/8:pi);  
quiver(x,y,sin(y),cos(x),'r')  
xlabel('x')  
ylabel('y')
```



Example: $\mathbf{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$$\mathbf{F}(x, y, z) = (x^2 - y^2, \cos 3y, z)$$

```
[x,y,z] = meshgrid(-2: .5 : 2, -2: .5 : 2, -2 : .5 | :2);  
quiver3(x,y,z,x.^2 - y.^2, cos(3 * y) ,z)  
xlabel('x')  
ylabel('y')  
zlabel('z')
```



Gradient Fields

A **Vector Field** is just a function F from \mathbb{R}^n to \mathbb{R}^n . A **Gradient Field** is a vector field which is the gradient of a real-valued function.

If f is a real-valued function of n variables such that $\nabla f = \mathbf{F}$, then f is called a **potential** of \mathbf{F} .

Gradient Fields

A **Gradient Field** is a vector field which is the gradient of a real-valued function.

The gradient $\nabla f(x, y)$ of $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$.

Example 1 $f(x, y) = (\cos x) \ln(1 + y^2) + \arctan(xy)$

$$\text{Here } \nabla f(x, y) = \left((-\sin x)(\ln(1 + y^2)) + \frac{y}{x^2y^2+1}, \frac{(-\cos x)(2y)}{1+y^2} + \frac{x}{x^2y^2+1} = (f_x(x, y), f_y(x, y)) \right)$$

Check that $f_{xy} = f_{yx}$ [Equality of Mixed Partial]

Example 2: Is $\mathbf{F}(x, y) = (2xy + \frac{y^2}{2}, \frac{x^2}{2} + 3xy)$ a gradient field? If $\mathbf{F} = \nabla f$, then

$$f_x(x, y) = 2xy + \frac{y^2}{2} \implies f_{xy}(x, y) = 2x + y$$

$$f_y(x, y) = \frac{x^2}{2} + 3xy \implies f_{yx}(x, y) = x + 3y$$

But these are not equal!

What if we try to build an f by "Partial Integration"?

$$f_x(x, y) = 2xy + \frac{y^2}{2} \implies f(x, y) = x^2y + x\frac{y^2}{2} + G(y) \implies$$
$$f_y(x, y) = x^2 + xy + G'(y) \text{ but we would need } G \text{ a function of } y$$

only such that $x^2 + xy + G'(y) = \frac{x^2}{2} + 3xy$; that is,

$$G'(y) = 2xy - \frac{x^2}{2}$$

Example: Find a potential function f if

$$\nabla f(x, y) = (2x \ln(xy) + x - y^3, \frac{x^2}{y} - 3y^2x)$$

Step 1: Check Equality Of Mixed Partial

$$f_x(x, y) = 2x \ln(xy) + x - y^3 \implies f_{xy} = 2x \frac{1}{xy} - 3y^2 = \frac{2x}{y} - 3y^2$$

$$f_y(x, y) = \frac{x^2}{y} - 3y^2x \implies f_{yx} = \frac{2x}{y} - 3y^2$$

Step 2: Integrate with respect to one of the variables

Here we will integrate f_y with respect to y so f has the form

$$f(x, y) = \int \frac{x^2}{y} - 3y^2x \, dy = x^2 \ln y - y^3x + H(x)$$

for some function H of x .

Step 3: Take partial derivative of the result of Step 2 with respect to the other variable to see how close we are to the result we want.

Fix the difference by adjusting the "constant" of integration.

With $f(x, y) = x^2 \ln y - y^3x + H(x)$, we have

$$f_x(x, y) = 2x \ln y - y^3 + H'(x)$$

With $f(x, y) = x^2 \ln y - y^3 x + H(x)$, we have

$$f_x(x, y) = 2x \ln y - y^3 + H'(x)$$

which we want equal to

$$2x \ln(xy) + x - y^3 = 2x \ln x + 2x \ln y + x - y^3$$

Thus we need $H'(x) = 2x \ln x + x$ so we can take

$$H(x) = x^2 \ln x + C$$

Step 4: Put it all together to form a potential function:

$$f(x, y) = x^2 \ln y - y^3 x + H(x) = x^2 \ln y - y^3 x + x^2 \ln x + C$$

Implicit Differentiation II

The Surface $2x^3y + yx^2 + t^2 = 0$ and the Plane $x + y + t - 1 = 0$

intersect along a Curve which contains the point
 $t = 1, x = -1, y = 1$

Check: Surface: $2(-1)(1) + 1(-1)^2 + 1^2 = 0$;

Plane: $-1 + 1 + 1 - 1 = 0$

Treat x and y as unknown functions of t .

Problem: Find $x'(t)$ and $y'(t)$ at $(t, x, y) = (1, -1, 1)$

Each equation defines a surface in 3-space and intersection of two surfaces is a curve.

The curve has some parametrization **G**

$$\mathbf{G}(t) = \begin{pmatrix} t \\ x(t) \\ y(t) \end{pmatrix}, \mathbb{R}^1 \rightarrow \mathbb{R}^3$$

$$\mathbf{G}(t) = \begin{pmatrix} t \\ x(t) \\ y(t) \end{pmatrix}, \mathbb{R}^1 \rightarrow \mathbb{R}^3$$

Consider $\mathbb{R}^1 \xrightarrow{\mathbf{G}} \mathbb{R}^3 \xrightarrow{\mathbf{F}} \mathbb{R}^2$

$$\text{where } \mathbf{F}(x, y, t) = \begin{pmatrix} F_1(t) \\ F_2(t) \end{pmatrix} = \begin{pmatrix} 2x^3y + yx^2 + t^2 \\ x + y + t - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Then $\mathbf{F}(\mathbf{G}(t)) = \mathbf{0}$ for all t

Differentiate using Chain Rule:

$$[\mathbf{F}(\mathbf{G}(t))]' = \mathbf{F}'(\mathbf{G}(t))\mathbf{G}'(t) = \begin{pmatrix} F_{1t} & F_{1x} & F_{1y} \\ F_{2t} & F_{2x} & F_{2y} \end{pmatrix} \begin{pmatrix} 1 \\ x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2t & 6x^2y + 2xy & 2x^3 + x^2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Write

$$\begin{pmatrix} 2t & 6x^2y + 2xy & 2x^3 + x^2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

as

$$\begin{pmatrix} 2t \\ 1 \end{pmatrix} + \begin{pmatrix} 6x^2y + 2xy & 2x^3 + x^2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$\begin{pmatrix} 6x^2y + 2xy & 2x^3 + x^2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = - \begin{pmatrix} 2t \\ 1 \end{pmatrix}$$

Multiply each side by inverse of coefficient matrix

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = - \begin{pmatrix} 6x^2y + 2xy & 2x^3 + x^2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2t \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = - \begin{pmatrix} 6x^2y + 2xy & 2x^3 + x^2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2t \\ 1 \end{pmatrix}$$

Evaluate at the given point: $t = 1, x = -1, y = 1$

$$\begin{aligned} \begin{pmatrix} x' \\ y' \end{pmatrix} &= - \begin{pmatrix} 6 - 2 & -2 + 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= - \begin{pmatrix} 4 & -1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= -\frac{1}{5} \begin{pmatrix} 1 & 1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= -\frac{1}{5} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -3/5 \\ -2/5 \end{pmatrix} \end{aligned}$$

More Generally

$\begin{cases} F_1(x, y, t) = 0 \\ F_2(x, y, t) = 0 \end{cases}$ define x, y implicitly as functions of t

Problem: Find $x'(t)$ and $y'(t)$ where $\mathbf{f}(t) = \begin{pmatrix} x \\ y \end{pmatrix}$.

Set Up: $\mathbb{R}^1 \xrightarrow{\mathbf{G}} \mathbb{R}^3 \xrightarrow{\mathbf{F}} \mathbb{R}^2$ where $\mathbf{G}(t) = \begin{pmatrix} t \\ x(t) \\ y(t) \end{pmatrix}$, $\mathbf{F}(t, x, y) = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$

Then $\mathbf{F}(\mathbf{G}(t)) \equiv 0$ so $\mathbf{F}'(\mathbf{G}(t))\mathbf{G}'(t) = 0$ which we write as

$$(F_t, F_x, F_y) \begin{pmatrix} 1 \\ x' \\ y' \end{pmatrix} = 0 \text{ or } F_t + [F_x, F_y][\mathbf{f}'(t)] = 0$$

$$\mathbf{f}'(t) = -[F_x, F_y]^{-1}F_t$$

Here the notation is

$$F_x = \begin{pmatrix} F_{1x} \\ F_{2x} \end{pmatrix}, F_y = \begin{pmatrix} F_{1y} \\ F_{2y} \end{pmatrix}, F_t = \begin{pmatrix} F_{1t} \\ F_{2t} \end{pmatrix}$$

IMPLICIT FUNCTION THEOREM

THEOREM: GIVEN A COLLECTION $F(\underline{x}, \underline{y}) = \underline{0}$ OF m EQUATIONS DEFINED IN TERMS OF \underline{x} (n VARIABLES) AND \underline{y} (m VARIABLES), SOLUTIONS TO $F(\underline{x}, \underline{y}) = \underline{0}$ NEAR A SOLUTION POINT $(\underline{x}, \underline{y}) = \underline{a}$ CAN BE REALIZED AS AN IMPLICIT FUNCTION

THE FINE PRINT: THE PARTIAL DERIVATIVES MUST EXIST AND BE CONTINUOUS

$$\underline{y} = \underline{y}(\underline{x}) \quad \text{IF} \quad \text{DET} \left[\frac{\partial F}{\partial \underline{y}} \right]_{\underline{a}} \neq 0$$

NOTE HOW THIS DERIVATIVE IS A SQUARE MATRIX...

THIS (LOCAL) SOLUTION IS UNIQUE AND DIFFERENTIABLE WITH

$$\left[\frac{\partial \underline{y}}{\partial \underline{x}} \right]_{\underline{a}} = - \left[\frac{\partial F}{\partial \underline{y}} \right]_{\underline{a}}^{-1} \left[\frac{\partial F}{\partial \underline{x}} \right]_{\underline{a}}$$

$$\frac{dy}{dx} = - \left(\frac{\partial F}{\partial x} \right) / \left(\frac{\partial F}{\partial y} \right)$$

COMPARE

Next Time

Maxima and Minima of Real-Valued Functions

