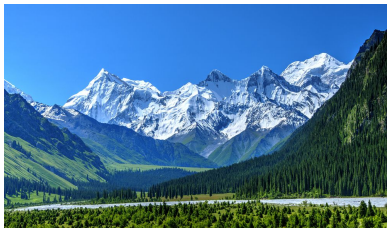


MATH 224: Vector Calculus

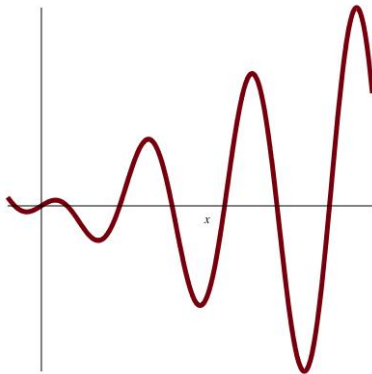


Class 17: March 18, 2026



Notes on Assignment 15
Assignment 16

Today:
Maxima and Minima of Real-Valued Functions

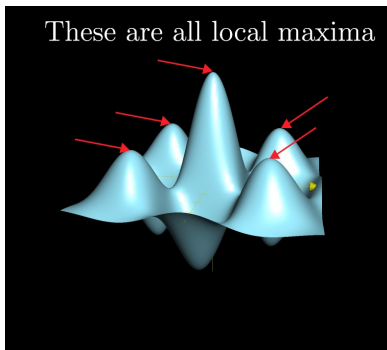


Let D be a subset of \mathbb{R}^n and $f : D \rightarrow \mathbb{R}^1$ be a real-valued function with \vec{x}_0 a point in D .

Definition: f has an **absolute maximum** at \vec{x}_0 if $f(\vec{x}_0) \geq f(\vec{x})$ for all \vec{x} in D .

Note: \geq makes sense because we are comparing real numbers.

f has a **relative maximum** at \vec{x}_0 if there is a neighborhood N around \vec{x}_0 such that $f(\vec{x}_0) \geq f(\vec{x})$ for all \vec{x} in N .



Theorem: Let \vec{x}_0 be an **interior** point of D . If f is differentiable at \vec{x}_0 and f has a relative maximum or minimum at \vec{x}_0 ,
then $f'(\vec{x}_0) = \nabla f(\vec{x}_0) = \vec{0}$.

Proof: Suppose f has a relative maximum at \vec{x}_0 .
Let \vec{u} be any unit vector in \mathbb{R}^n .

$$\text{Then } \frac{\partial f}{\partial \vec{u}} = \lim_{t \rightarrow 0} \frac{f(\vec{x}_0 + t\vec{u}) - f(\vec{x}_0)}{t}$$

$$\text{(a) Take } \lim_{t \rightarrow 0^+} : \frac{-}{+} \leq 0$$

$$\text{thus } \frac{\partial f}{\partial \vec{u}} = 0 \text{ for all } \vec{u}$$

$$\text{(b) Take } \lim_{t \rightarrow 0^-} : \frac{-}{-} \geq 0$$

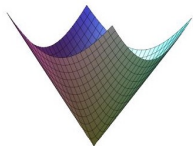
Taking \vec{u} to be unit vectors gives $\nabla f(\vec{x}_0) = \vec{0}$

Theorem: f differentiable at relative extrema implies gradient is 0.

The Theorem Has Its Limitations:

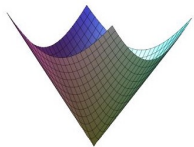
(1) The function can have an extreme value at a point where it is not differentiable.

Example: $f(x, y) = \sqrt{x^2 + y^2}$ has minimum at $(0,0)$ but is not differentiable there.



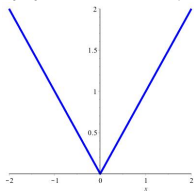
$$f_x(x, y) = \frac{x}{\sqrt{x^2 + y^2}}, f_y(x, y) = \frac{y}{\sqrt{x^2 + y^2}},$$

Example: $f(x, y) = \sqrt{x^2 + y^2}$ has minimum at $(0,0)$ but is not differentiable there.



Analogue in Calculus I:

$$f(x) = \sqrt{x^2} = |x|$$

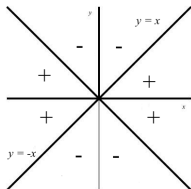
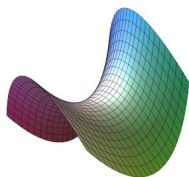


Theorem: f differentiable at relative extrema implies gradient is 0.

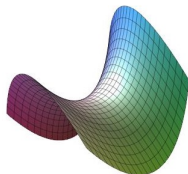
The Theorem Has More Limitations:

(2) We can have $\nabla f(\vec{x}_0) = 0$ but no extreme point at \vec{x}_0

$$f(x, y) = x^2 - y^2$$



$$\nabla f(x, y) = (2x, -2y)$$



There is a Maximum in one direction and a Minimum in another

Saddle Point



Quiz:
Name a Famous
Commercial Food Product
That Exhibits
a Saddle Point



Definition: A point \vec{x}_0 in the domain of f is a **Critical Point** of f if

$$(a) \nabla f(\vec{x}_0) = \vec{0}$$

or

(b) ∇f does not exist at \vec{x}_0 .

**Extreme Values Can Occur at Critical Points
or Points on the Boundary**

Example: Temperature Distribution on disk of radius 1 centered at origin is $T(x, y) = 2x^2 + 4y^2 + 2x + 1$.

For Critical Points, examine $\nabla T = (4x + 2, 8y)$

$\nabla T = (0, 0)$ only at $x = -\frac{1}{2}, y = 0$

which does lie inside the disk.

Note $T(-\frac{1}{2}, 0) = 2(\frac{1}{4}) + 4(0^2) + 2(-\frac{1}{2}) + 1 = \frac{1}{2}$, and $T(0, 0) = 1$.

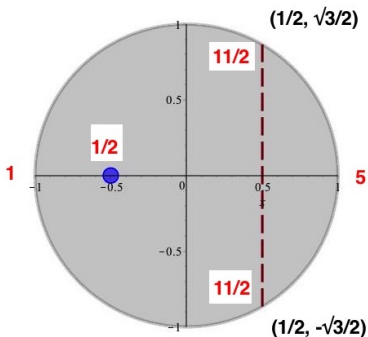
Analyze Along Boundary:

$$x^2 + y^2 = 1 \text{ so } y^2 = 1 - x^2 \text{ and}$$

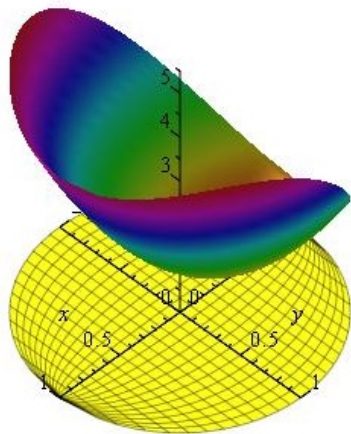
$$T(x, y) = g(x) = 2x^2 + 4(1 - x^2) + 2x + 1 = -2x^2 + 2x + 5$$

Thus $g'(x) = -4x + 2$, $g''(x) = -4$ so $x = \frac{1}{2}$ gives a maximum.

$$x = \frac{1}{2} \text{ gives } y^2 = 1 - \frac{1}{4} = \frac{3}{4} \text{ so } y = \pm \frac{\sqrt{3}}{2}$$



red numbers are values of the function



Parametrize Boundary

$$x = \cos t, y = \sin t \text{ for } 0 \leq t \leq 2\pi$$

$$\begin{aligned} T(x, y) &= 2x^2 + 4y^2 + 2x + 1 \\ &= 2\cos^2 t + 4\sin^2 t + 2\cos t + 1 \\ &= 2\cos^2 t + 2\sin^2 t + 2\sin^2 t + 2\cos t + 1 \\ &= 2 + 2\sin^2 t + 2\cos t + 1 = 2\sin^2 t + 2\cos t + 3 \\ &= H(t) \end{aligned}$$

$$H(0) = 2 \cdot 1 + 2 \cdot 0 + 3 = 5, H(\pi) = 2 \cdot 1 + 2 \cdot (-1) + 3 = 1$$

Now $H'(t) = 4\sin t \cos t - 2\sin t = 2\sin t(2\cos t - 1)$ so

$$H'(t) = 0 \text{ at } \sin t = 0 \text{ or } \cos t = \frac{1}{2}$$

The first condition gives $t = 0, t = \pi$, the second occurs when

$$t = \frac{\pi}{3}.$$

Next Time:

Solving Constrained Optimization Problems Using Lagrange Multipliers

Joseph-Louis Lagrange



As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection.

AZ QUOTES