

MATH 224: Vector Calculus

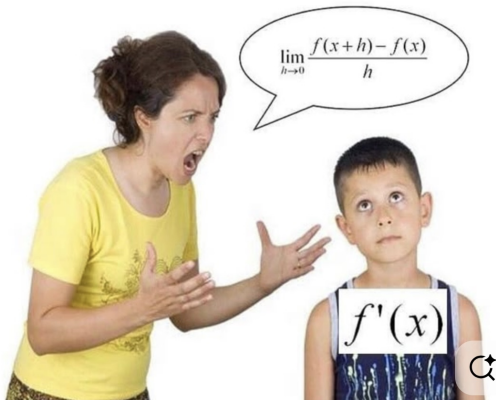
Notes on Class 2

February 11, 2026



MATLAB Workshops

When your mom calls you by your full name:



Definition of Derivative

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \text{ IF THE LIMIT EXISTS}$$

Example: Find $f'(x)$ if $f(x) = x^3$

Solution:

$$\begin{aligned} f(x+h) - f(x) &= (x+h)^3 - x^3 \\ &= x^3 + 3x^2h + 3xh^2 + h^3 - x^3 \\ &= 3x^2h + 3xh^2 + h^3 \\ &= h [3x^2 + 3xh + h^2] \end{aligned}$$

$$\text{so } \frac{f(x+h) - f(x)}{h} = [3x^2 + 3xh + h^2] \text{ if } h \neq 0$$

$$\text{Hence } f'(x) = \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2) = 3x^2$$

Why is Division by Zero Undefined?

$$a/b = a \div b$$

Basic Arithmetic: Addition (+) and Multiplication (*)

We define subtraction in terms of addition: $a - b = c$ means c is the unique number so that $a = b + c$

Similarly, we define division in terms of multiplication: $a/b = c$ means c is the unique number so that $a = bc$

Analog of Straight Line In Higher Dimensions

Line: $ax + by = c$

$$a_1x_1 + a_2x_2 = c$$

Plane: $a_1x_1 + a_2x_2 + a_3x_3 = d$

$$ax + by + cz = d$$

Hyperplane: $a_1x_1 + a_2x_2 + \dots + a_nx_n = d$

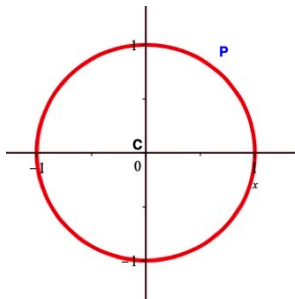
Other Important Curves: CIRCLES and ELLIPSES
and the counterparts in higher dimensions.

Recall: Graph of $f : \mathcal{R}^1 \rightarrow \mathcal{R}^1$ is a curve in the plane.

**BUT: NOT EVERY CURVE IN THE PLANE IS THE
GRAPH OF SUCH A FUNCTION**

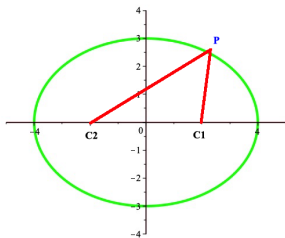
Vertical Line Test

CIRCLE



Set of all points a
fixed distance from
a fixed point (center)
 $\text{distance}(P, C) = r$

ELLIPSE



Set of all points,
sum of distances
to pair of fixed points is constant
 $d(P, C_1) + d(P, C_2) = r$

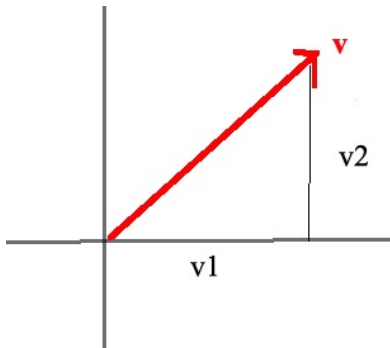
Distance in \mathcal{R}^n

Magnitude of a vector $\mathbf{v} = |\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$

Where does this come from?

Consider $\mathbf{v} = (v_1, v_2)$ in \mathcal{R}^2 where $\mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2$

$\sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2}$ (Pythagorean Theorem)



Distance Between x and $y = |x - y|$

Examine Circle in the Plane

Center $\mathbf{C} = (a, b)$ and radius r

Variable Point $\mathbf{P} = (x, y)$

Defining Relationship: $d(\mathbf{P}, \mathbf{C}) = r$

which means

$$|\mathbf{P} - \mathbf{C}| = r$$

$$|(x - a, y - b)| = r$$

$$\sqrt{(x - a)^2 + (y - b)^2} = r$$

$$(x - a)^2 + (y - b)^2 = r^2$$

Multiply Out: $x^2 - 2ax + a^2 + y^2 - 2by + b^2 = r^2$

$$x^2 + y^2 - 2ax - 2by = r^2 - a^2 - b^2$$

which has the form $x^2 + y^2 + Ax + By = C$.

Can We Go Backwards?

Example: $x^2 + y^2 - 6x + 16y = 71$

Complete the Squares in x and y

$$(x^2 - 6x) + (y^2 + 16y) = 71$$

$$(x^2 - 6x + 9) + (y^2 + 16y + 64) = 71 + 9 + 64$$

$$(x - 3)^2 + (y + 8)^2 = 144 = 12^2$$

Circle as Image of a function $f : \mathcal{R}^1 \rightarrow \mathcal{R}^2$

Parametrization with parameter t

Example: $\mathbf{f}(t) = (\cos t, \sin t), 0 \leq t \leq 2\pi$

Example:

$$\begin{cases} x = 12 \cos t + 3 \\ y = 12 \sin t - 8 \end{cases}$$

$$\begin{cases} x - 3 = 12 \cos t \\ y + 8 = 12 \sin t \end{cases}$$

$$(x - 3)^2 + (y + 8)^2 = 12^2$$

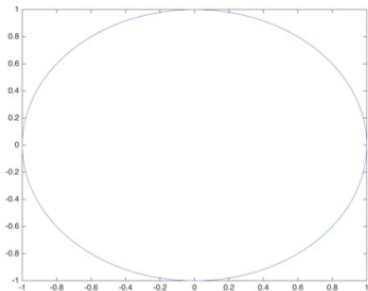
$$\mathbf{f}(t) = (12 \cos t + 3, 12 \sin t - 8)$$

Plotting Parametric Curves in MATLAB

Unit Circle

$$x^2 + y^2 = 1$$

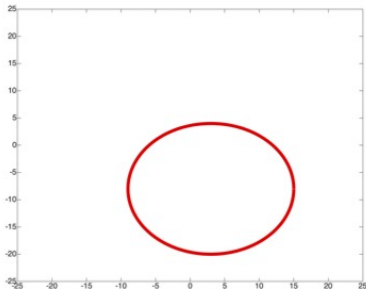
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t = [0: .01: 2*pi];  
plot(cos(t),sin(t))
```



Circle of Radius 12 and Center at (3,-8)

$$(x - 3)^2 + (y + 8)^2 = 12^2$$

```
t = [0: .01 : 2*pi];  
plot(12 * cos(t) + 3, 12*sin(t)-8, 'r', 'LineWidth',4)  
xlim([-25,25])  
ylim([-25,25])
```



ELLIPSE

Standard Ellipse:

Center at $(0,0)$

Foci at $(\pm c, 0)$

Vertices $(\pm a, 0)$ and $(0, \pm b)$

$(a,0)$ distance from $(c,0)$ + distance from $(-c,0)$

$$(a - c) + (a - (-c)) = a - c + a + c = 2a$$

$(0,b)$ distance from $(c,0)$ + distance from $(-c,0)$

$$\sqrt{c^2 + b^2} + \sqrt{c^2 + b^2} = 2\sqrt{c^2 + b^2}$$

$$\text{Thus } 2\sqrt{c^2 + b^2} = 2a$$

$$\text{So } c^2 + b^2 = a^2 \text{ implying } c^2 = a^2 - b^2$$

(x,y) distance from $(c,0)$ + distance from $(-c,0) = 2a$

$$\sqrt{(x - c)^2 + y^2} + \sqrt{(x + c)^2 + y^2} = 2a$$

Much algebra yields

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Parametrization: $\mathbf{g}(t) = (a \cos t, b \sin t), 0 \leq t \leq 2\pi$

The Algebra

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a$$

Write as $\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$

Square Both Sides

$$(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2$$

Expand, Simplify, and Divide by 4

$$x^2 + 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + x^2 - 2cx + c^2 + y^2$$

$$4cx - 4a^2 = -4a\sqrt{(x-c)^2 + y^2}$$

$$cx - a^2 = -a\sqrt{(x-c)^2 + y^2}$$

Begin with $cx - a^2 = -a\sqrt{(x - c)^2 + y^2}$

Square Again

$$c^2x^2 - 2a^2cx + a^4 = a^2((x - c)^2 + y^2)$$

$$c^2x^2 - 2a^2cx + a^4 = a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2$$

$$c^2x^2 + a^4 = a^2x^2 + a^2c^2 + a^2y^2$$

Write as $(a^2 - c^2)x^2 + a^2y^2 = a^4 - a^2c^2 = a^2(a^2 - c^2)$

But $a^2 - c^2 = b^2$ so

$$b^2x^2 + a^2y^2 = a^2b^2$$

Divide by a^2b^2 :

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$