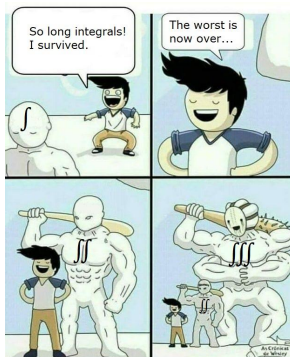
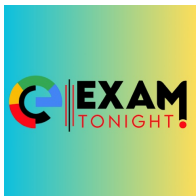


# MATH 224: Vector Calculus



Class 22: Monday April 6, 2026



Tonight 7 PM –?  
Warner 105  
One Sheet of Notes

# Department of Mathematics and Statistics

## Pre-registration Dessert Social

Wednesday, 4/15 | 3:30-4:30pm | Warner 105

Interested in taking some Math or Stat courses in **Fall 2026**? Currently taking a Math or Stats class? Need a study break?



Join the Math & Stats faculty over dessert to:

- Learn about Fall 2026 course offerings
- Get information about:
  - Major in Mathematics and/or the Applied Math Track
  - Major in Statistics
  - Minor in Mathematics
- Ask questions and receive advice about how Math and Stats fits into your Middlebury experience
- Be in community and hear from other students about Math and Stats courses

**Anyone who is currently taking or wants to take a Math or Stats course is welcome! Even if you're graduating in May, we hope to see you at the dessert social!**



## Multiple Integrals

## The Week Ahead:

Iterated Integral (Last Time)  
Definition of Multiple Integrals  
Properties of the Integral  
Change of Variable

## Instances of the Integral: I

### The Classic Case

$$f := \mathbb{R}^1 \rightarrow \mathbb{R}^1$$

Definite Integral  $\int_a^b f(x) dx$

Indefinite Integral  $\int f(x) dx$

## Instances of the Integral: II

### Vector-Valued Functions of a Real Variable

$$f := \mathbb{R}^1 \rightarrow \mathbb{R}^n$$

$$f(t) = [f_1(t), f_2(t), \dots, f_n(t)]$$

$$\int f(t) = \left[ \int f_1(t), \int f_2(t), \dots, \int f_n(t) \right]$$

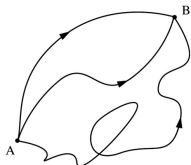
## Instances of the Integral: III

Line Integral = Path Integral  
(Will Study in Chapter 7)

$\gamma$  is graph of  $g : \mathbb{R}^1 \rightarrow \mathbb{R}^n, a \leq t \leq b$

Force Field  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$

$$\int_{\gamma} F = \int_a^b F(g(t)) \cdot g'(t) dt$$

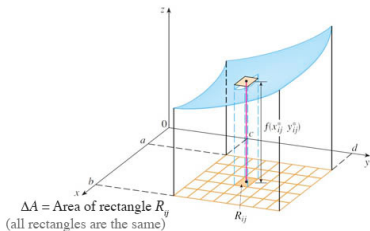
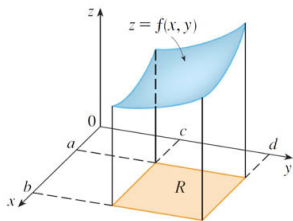


## Instances of the Integral: IV

### Iterated Integral

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}^1$$

$$\int_a^b \left( \int_{y=g_1(x)}^{y=g_2(x)} f(x, y) dy \right) dx$$



## Instances of the Integral: V

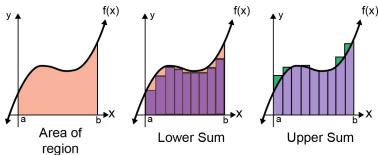
### The Multiple Integral

#### A Generalization of Situation 1

$$\int_a^b f(x)dx = \lim_{\max(\Delta(x_i)) \rightarrow 0} \sum_{i=1}^n f(x_i)\Delta x_i$$

$\Delta x_i$  = length of  $i$ th subdivision

$x_i$  = any point in  $i$ th subinterval



## Instances of the Integral: V

### Unequal Divisions

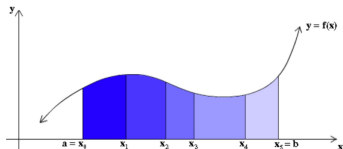
$$\int_a^b f(x) dx = \lim_{\max(\Delta(x_i)) \rightarrow 0} \sum_{i=1}^n f(x_i) \Delta x_i$$

$\Delta x_i$  = length of  $i$ th subdivision

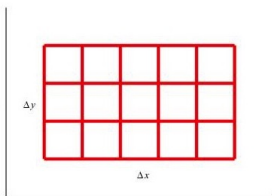
$x_i$  = any point in  $i$ th subinterval

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### Riemann Sums of Unequal Length Subintervals



First Extension:  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^1$  on a rectangle  $\mathcal{R}$



Cover  $\mathcal{R}$  with a grid  $G$  of horizontal and vertical lines  
 $mesh(G) = m(G)$  = maximum length of edge of interval in grid.

Number Rectangles  $R_1, R_2, \dots, R_k$  with area of Rectangle  $R_i$   
denoted by  $A(R_i)$ .

Pick a point  $\vec{x}_i$  in  $R_i$ .

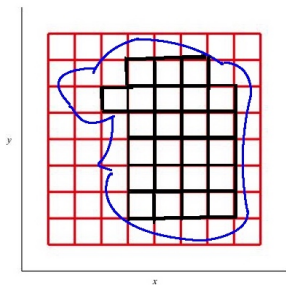
$$\text{Form } \sum_{i=1}^k f(\vec{x}_i)A(R_i); \text{ Take } \lim_{m(G) \rightarrow 0} \sum_{i=1}^k f(\vec{x}_i)A(R_i)$$

The limit is the integral of  $f$  over  $\mathcal{R}$  and is denoted  $\int_{\mathcal{R}} f dA$

Second Extension:  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^1$  on a BOUNDED set  $\mathcal{B}$

Cover  $\mathcal{B}$  with a grid  $G$  of horizontal and vertical lines

Let  $R_1, R_2, \dots, R_k$  be all bounded rectangles formed by  $G$  that lie **inside**  $\mathcal{B}$ .



Choose  $\vec{x}_i$  in  $R_i$ .

$$\text{Take } \lim_{m(G) \rightarrow 0} \sum_{i=1}^k f(\vec{x}_i) A(R_i)$$

The limit is the integral of  $f$  over  $\mathcal{B}$  and is denoted  $\int_{\mathcal{B}} f dV$

Theorem: If  $\int_B f dV$  exists and iterated integrals exist for some orders of partial integration, then all of these integrals are equal.

Proof:

Apostol, *Mathematical Analysis*

Sprivak, *Calculus on Manifolds*



## When Does The Integral Exist?

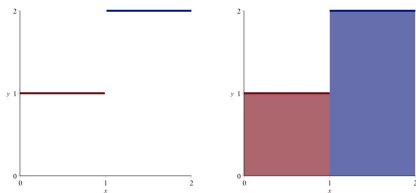
Idea:  $f$  does not have too many points of discontinuity

Definition: A set  $S$  has **zero content** if  $\int_S 1dV = 0$ .

Theorem: Let  $\mathcal{B}$  be a bounded set in  $\mathbb{R}^n$  whose boundary has zero content. Let  $f$  be a bounded function bounded on  $\mathcal{B}$ . If  $f$  is continuous on  $\mathcal{B}$  except perhaps on a set of zero content, then  $\int_{\mathcal{B}} fdV$  exists.

## Example from Calculus 1

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1, \\ 2 & 1 < x \leq 2 \end{cases}$$



$$\int_1^2 f(x) dx = 3$$

Generalize For a Function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^1$

Coordinate Rectangle in  $\mathbb{R}^n$

$$\mathcal{R} = \{(x_1, x_2, \dots, x_n) : a_1 \leq x_1 \leq b_1, a_2 \leq x_2 \leq b_2, \dots, a_n \leq x_n \leq b_n\}$$

Volume or Content of  $\mathcal{R}$

$$V(\mathcal{R}) = (b_1 - a_1)(b_2 - a_2) \dots (b_n - a_n)$$

Grid: a finite set of  $n - 1$  =dimensional planes in  $\mathbb{R}$  parallel to the coordinate planes.

$G$  divides  $\mathbb{R}$  into a finite number of bounded "rectangles"

$\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_k$  and possibly other unbounded "rectangles,

The mesh  $m(G)$  of a grid = maximum "length" of a side of the rectangles  $\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_k$

A set  $\mathcal{B}$  is bounded if it can be covered by a grid.

$$\text{Then } \int_{\mathcal{B}} f dV = \lim_{m(G) \rightarrow 0} \sum_{i=1}^k f(\vec{x}_i) V(R_i)$$

if this limit exists. for all grids and all choices of  $\vec{x}_i$  in  $R_i$ .

$$\text{Then } \int_{\mathcal{B}} f dV = \lim_{m(G) \rightarrow 0} \sum_{i=1}^k f(\vec{x}_i) V(R_i)$$

if this limit exists. for all grids and all choices of  $\vec{x}_i$  in  $R_i$ .

$$\text{Content of } \mathcal{B} = \int_{\mathcal{B}} 1 dV = \begin{cases} \text{Length of } \mathcal{B} & \text{if } \mathcal{B} \subset R^1, \\ \text{Area of } \mathcal{B} & \text{if } \mathcal{B} \subset R^2 \\ \text{Volume of } \mathcal{B} & \text{if } \mathcal{B} \subset R^3 \end{cases}$$

Example Evaluate  $\int_{\mathcal{B}}(x^2 + 5y)dV$  where  $0 \leq x \leq 1, 0 \leq y \leq 3$   
**using the definition.**

The existence of the integral is guaranteed since  $\mathcal{B}$  is bounded and  
 $f(x, y) = x^2 + 5y$  is continuous on  $\mathcal{B}$

Hence any sequence of Riemann sums with mesh going to 0 can be  
used.

For each  $n = 1, 2, \dots$  consider the Grid  $G_n$  consisting of  
the vertical lines  $x = \frac{i}{n}, i = 0, 1, \dots, n$  and  
the horizontal lines  $y = \frac{j}{n}, j = 0, 1, \dots, 3n$

Then mesh of  $G_n = \frac{1}{n}$  and Area of Rectangle  $R_{ij} = \frac{1}{n^2}$

Riemann sum is  $\sum_{i=1}^n \left( \sum_{j=1}^{3n} \left[ \left( \frac{i}{n} \right)^2 + 5 \left( \frac{j}{n} \right) \right] \right) A(R_{ij})$

$$= \frac{1}{n^2} \left[ \sum_{i=1}^n \sum_{j=1}^{3n} \left( \frac{i}{n} \right)^2 + \sum_{i=1}^n \sum_{j=1}^{3n} 5 \left( \frac{j}{n} \right) \right]$$

$$= \frac{1}{n^2} \left[ 3n \sum_{i=1}^n \left( \frac{i}{n} \right)^2 + n \sum_{j=1}^{3n} \frac{5j}{n} \right]$$

$$= \frac{1}{n^2} \left[ \frac{3n}{n^2} \sum_{i=1}^n i^2 + \frac{5n}{n} \sum_{j=1}^{3n} j \right]$$

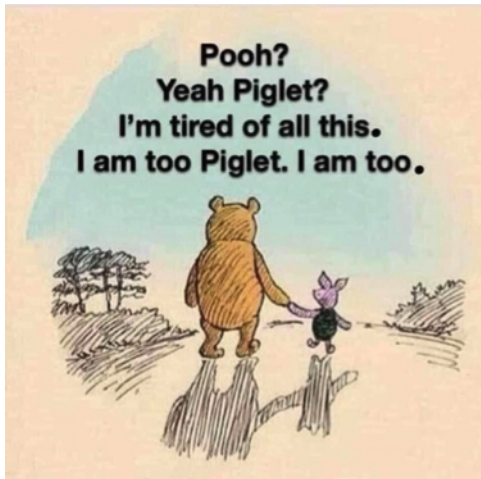
$$= \frac{1}{n^2} \left[ \frac{3}{n} \frac{n(n+1)(2n+1)}{6} + 5 \frac{(3n)(3n+1)}{2} \right]$$

$$\text{Riemann sum is } \sum_{i=1}^n \left( \sum_{j=1}^{3n} \left[ \left(\frac{i}{n}\right)^2 + 5\left(\frac{j}{n}\right) \right] \right) A(R_{ij})$$

$$\begin{aligned} &= \frac{1}{n^2} \left[ \frac{1}{2}(n+1)(2n+1) + \frac{15}{2}n(3n+1) \right] \\ &= \frac{1}{2} \left[ \left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) \right] + \frac{15}{2} \left[ 3 + \frac{1}{n} \right] \end{aligned}$$

$$\text{Hence } \lim_{n \rightarrow \infty} = \frac{1}{2}(2) + \frac{15}{2}(3) = \frac{47}{2}$$

**Pooh?  
Yeah Piglet?  
I'm tired of all this.  
I am too Piglet. I am too.**



**There Must Be a Better  
Way!**

# Evaluate As Iterated Integral

$$\int_{x=0}^{x=1} \int_{y=0}^{y=3} (x^2 + 5y) dy dx$$

$$= \int_{x=0}^{x=1} \left[ x^2 y + \frac{5}{3} y^2 \right]_{y=0}^{y=3} dx$$

$$= \int_0^1 3x^2 + \frac{45}{2} dx = \left[ x^3 + \frac{45}{2} x \right]_0^1 = \left( 1 + \frac{45}{2} \right) - (0 + 0) = \frac{47}{2}$$