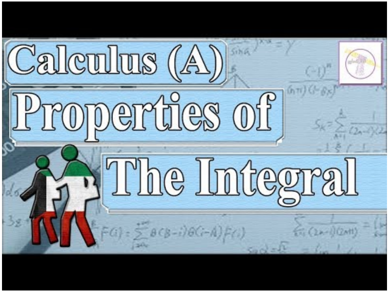
MATH 224: Vector Calculus



Class 23: Wednesday, April 8, 2026

Department of Mathematics and Statistics

Pre-registration Dessert Social

Wednesday, 4/15 | 3:30-4:30pm | Warner 105

Interested in taking some Math or Stat courses in **Fall 2026?** Currently taking a Math or Stats class? Need a study break?

Join the Math & Stats faculty over dessert to:

- Learn about Fall 2026 course offerings
- Get information about:
  - Major in Mathematics and/or the Applied Math Track
  - Major in Statistics
  - Minor in Mathematics
- Ask questions and receive advice about how Math and Stats fits into your Middlebury experience
- Be in community and hear from other students about Math and Stat courses

Anyone who is currently taking or wants to take a Math or Stats course is welcome! Even if you're graduating in May, we hope to see you at the dessert social!

Handouts

Multiple Integrals: Integration Theorems

Announcements

Review: Change of Variable (Method of Substitution)
Improper Integrals

This Week:
Definition of Multiple Integrals (Last Time)
Properties of the Integral
Change of Variable
Application to Probability

Example Evaluate $\int_{\mathcal{B}}(x^2 + 5y)dV$ where $0 \leq x \leq 1, 0 \leq y \leq 3$

using the definition

The existence of the integral is guaranteed since $\mathcal{B}$ is bounded and $f(x, y) = x^2 + 5y$ is continuous on $\mathcal{B}$

Hence any sequence of Riemann sums with mesh going to 0 can be used.

For each $n = 1, 2, ...$ consider the Grid G_n consisting of

the vertical lines $x = \frac{i}{n}, i = 0, 1, ..., n$ and

the horizontal lines $y = \frac{j}{n}, j = 0, 1, ..., 3n$

Then mesh of $G_n = \frac{1}{n}$ and Area of Rectangle $R_{ij} = \frac{1}{n^2}$

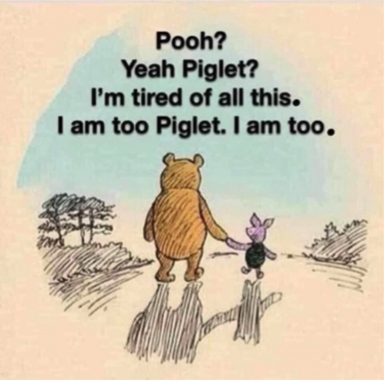
Riemann sum is $\displaystyle\sum_{i=1}^{n}\left(\sum_{j=1}^{3n}\left[\left(\frac{i}{n}\right)^2+5\left(\frac{j}{n}\right)\right]\right)A(R_{ij})$

$$\begin{aligned}
&= \frac{1}{n^2}\left[\sum_{i=1}^{n}\sum_{j=1}^{3n}\left(\frac{i}{n}\right)^2+\sum_{i=1}^{n}\sum_{j=1}^{3n}5\left(\frac{j}{n}\right)\right]\\
&= \frac{1}{n^2}\left[3n\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2+n\sum_{j=1}^{3n}\frac{5j}{n}\right]\\
&= \frac{1}{n^2}\left[\frac{3n}{n^2}\sum_{i=1}^{n}i^2+\frac{5n}{n}\sum_{j=1}^{3n}j\right]\\
&= \frac{1}{n^2}\left[\frac{3}{n}\frac{n(n+1)(2n+1)}{6}+5\frac{(3n)(3n+1)}{2}\right]
\end{aligned}$$

Riemann sum is $\displaystyle\sum_{i=1}^{n}\left(\sum_{j=1}^{3n}\left[(\frac{i}{n})^2+5(\frac{j}{n})\right]\right)A(R_{ij})$

$$\begin{aligned}
&= \frac{1}{n^2}\left[\frac{1}{2}(n+1)(2n+1)+\frac{15}{2}n(3n+1)\right] \\
&= \frac{1}{2}\left[(1+\frac{1}{n})(2+\frac{1}{n})\right]+\frac{15}{2}\left[3+\frac{1}{n}\right]
\end{aligned}$$

Hence $\displaystyle\lim_{n\to\infty}=\frac{1}{2}(2)+\frac{15}{2}(3)=\frac{47}{2}$



There Must Be a Better Way!

Evaluate As Iterated Integral

$$\int_{x=0}^{x=1}\int_{y=0}^{y=3}(x^2+5y)dydx$$

$$=\int_{x=0}^{x=1}\left[x^2y+\frac{5}{3}y^2\right]_{y=0}^{y=3}dx$$

$$=\int_0^1 3x^2+\frac{45}{2}dx=\left[x^3+\frac{45}{2}x\right]_0^1=(1+\frac{45}{2})-(0+0)=\frac{47}{2}$$

MULTIPLE INTEGRAL

Definition A function f is **integrable** over a bounded set $\mathcal{B}$ if there is a number $\int_{\mathcal{B}} f dV$ such that $\lim_{mesh(g)\to 0} \sum f(\vec{x_i})v(R_i)) = \int_{\mathcal{B}} f dV$ for every grid G covering $\mathcal{B}$ with mesh (G) and any choice of $\vec{x_i}$ in $\mathcal{R}_{\rangle}$

What This Limit Statement Means: For every $\epsilon > 0$, there is a $\delta > 0$ such that if G is a grid of mesh $< \delta$, then $|\int_{\mathcal{B}} f dV - \sum f(\vec{x_i})v(R_i))| < \epsilon$.

Theorem (not proved): $\int_{\mathcal{B}} f dV$ can be evaluated by Iterated Integrals.

Properties of the Integral

Linearity

Suppose f and g are both integrable over $\mathcal{B}$ while a and b are any real numbers.

Then $af + bg$ is integrable over $\mathcal{B}$ and
$\int_{\mathcal{B}}(af + bg)dV = a\int_{\mathcal{B}} fdV + b\int_{\mathcal{B}} gdV$

Corollary (1) The set $\mathcal{V}$ of functions integrable over $\mathcal{B}$ is closed under addition and scalar multiplication so $\mathcal{V}$ is a vector space.
(2) The function $L : \mathcal{V} \to \mathbb{R}^1$ given by $L(f) = \int_{\mathcal{B}} fdV$ is a linear transformation.

Let $\epsilon > 0$ be given. Choose $\delta > 0$ so that if S_1 and S_2 are Riemann sums for f and g respectively with mesh $< \delta$, then $|a||S_1 - \int_{\mathcal{B}} f dV| < \frac{\epsilon}{2}$ and $|b||S_2 - \int_{\mathcal{B}} g dV| < \frac{\epsilon}{2}$.

Now let S be a Riemann sum for $af + bg$ with mesh of grid $< \delta$.

$$\begin{aligned}
\text{Then } S &= \sum (af + bg) f(\vec{x_i}) V(R_i) \\
&= a \sum f(\vec{x_i}) V(R_i) + b \sum g(\vec{x_i}) V(R_i) \\
&= aS_1 + bS_2
\end{aligned}$$

$$\text{Now } |S - a \int f dV - b \int g dV| = |aS_1 - a \int f dV + bS_2 - b \int g dV|$$
$$\leq |a||S_1 - \int f dV| + |b||S_2 - \int g dV| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Theorem:(**Positivity**) If f is nonnegative and integrable over $\mathcal{B}$, then $\int_{\mathcal{B}} f dV \geq 0$.

Theorem: If f, g are integrable on $\mathcal{B}$ with $f \geq g$, then $\int f \geq \int g$.

Proof: $(f - g) \geq 0$ implies $\int_{\mathcal{B}} (f - g) dV \geq 0$

so $0 \leq \int_{\mathcal{B}} (f - g) dV = \int_{\mathcal{B}} f dV - \int_{\mathcal{B}} g dV$

Hence $\int_{\mathcal{B}} f dV \geq \int_{\mathcal{B}} g dV$

Theorem: If f and $|f|$ are integrable over $\mathcal{B}$, then

$$|\int_{\mathcal{B}} f dV| \leq \int_{\mathcal{B}} |f| dV$$

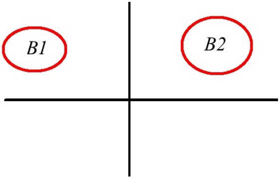
Proof: Start with $-|f| \leq f \leq |f|$

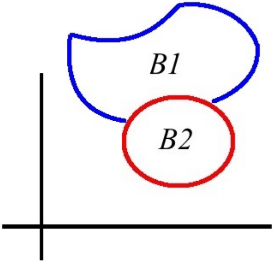
Then $-\int_{\mathcal{B}} |f| \leq \int_{\mathcal{B}} f \leq \int_{\mathcal{B}} |f|$

So $|\int_{\mathcal{B}} f| \leq \int_{\mathcal{B}} |f|$

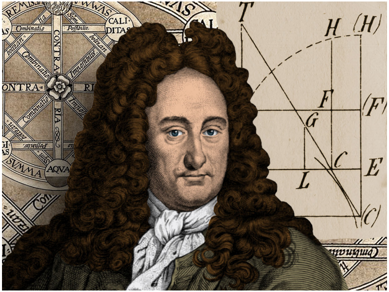
Theorem (**Additivity**): If f is integrable over disjoint sets $\mathcal{B}_1$ and $\mathcal{B}_2$, then f is integrable over $\mathcal{B}_1 \cup \mathcal{B}_2$ with

$$\int_{\mathcal{B}_1 \cup \mathcal{B}_2} f = \int_{\mathcal{B}_1} f + \int_{\mathcal{B}_2} f$$





Leibniz Rule



Gottfried Wilhelm von Leibniz

July 1, 1646 – November 14, 1716

Biography

Leibniz Rule: Interchanging Differentiation and Integration

If g_y is continuous on $a \leq x \leq b, c \leq y \leq d$, then

$$\frac{d}{dy}\int_a^b g(x,y)dx = \int_a^b \frac{\partial}{\partial y} g(x,y)dx$$

$$\frac{d}{dy}\int_a^b g(x,y)dx = \int_a^b \frac{\partial}{\partial y} g(x,y)dx$$

Example Compute $f(x) = \int_0^1 \frac{u^x - 1}{\ln u} du$
By Leibniz:

$$f'(x) = \int_0^1 \frac{1}{\ln u}(u^x \ln u)du = \int_0^1 u^x du = \frac{u^{x+1}}{x+1}\bigg|_{u=0}^{u=1} = \frac{1}{x+1}$$

So $f(x) = \ln(x+1) + C$ for some constant C.
To Find C, evaluate at $x = 0$:
$f(0) = \int_0^1 \frac{u^0 - 1}{\ln u} du = \int_0^1 0 = 0$
But $f(0) = \ln(0+1) + C = \ln(1) + C = 0 + C = C$ so $C = 0$ and
$f(x) = \ln(x+1)$

Example: Find $f'(y)$ if $f(y) = \int_0^1 (y^2 + t^2)dt$

Method I: $f(y) = \int_0^1 (y^2 + t^2)dt = (y^2 t + \frac{t^3}{3})\Big|_{t=0}^{t=1} = y^2 + \frac{1}{3}$ so

$$f'(y) = 2y$$

Method II: (Leibniz) $f'(y) = \int_0^1 2y dt = 2yt\Big|_0^1 = 2y$

Proof of Leibniz Rule

To Show:

$$\frac{d}{dy}\int_a^b g(x,y)dx = \int_a^b \frac{\partial}{\partial y} g(x,y)dx$$

Let $f(y) = \int_a^b g(x,y)dx$ and Use Definition of Derivative

$$f'(y) = \lim_{h\to 0} \frac{f(y+h)-f(y)}{h}$$

$$\frac{f(y+h)-f(y)}{h} = \frac{\int_a^b g(x,y+h)dx - \int_a^b g(x,y)dx}{h} = \frac{\int_a^b (g(x,y+h)-g(x,y))dx}{h}$$

$$f'(y) = \lim_{h \to 0} \frac{f(y+h) - f(y)}{h} = \lim_{h \to 0} \frac{\int_a^b [g(x, y+h) - g(x,y)]\, dx}{h}$$

Interchange Limit and Integral:

$$= \int_a^b \left(lim_{h \to 0} \frac{[g(x, y+h) - g(x,y)]}{h} \right) dx$$

$$= \int_a^b \frac{\partial g}{\partial y}(x,y) dx$$

Alternate Proof of Leibniz Rule

(**Uses Iterated Integral**)

Begin with $\int_a^b g_y(x,y)dx$

Let $I = \int_c^y (\int_a^b g_y(x,y)dx)dy$

Switch Order of Integration: $I = \int_a^b \left(\int_c^y g_y(x,y)dy\right) dx$

$$I = \int_a^b g(x,y)\Big|_{y=c}^{y=y} dx = \int_a^b g(x,y) - g(x,c)dx$$

$$= \int_a^b g(x,y)dx - \int_a^b g(x,c)dx$$

The left term is a function of y and the second is a constant C

Alternate Proof of Leibniz Rule (Continued)

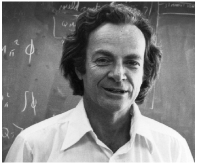
$$I = \int_c^y \left(\int_a^b g_y(x,y)dx \right) dy = \int_a^b g(x,y)dx - C$$

Now Take the Derivative of Each Side with Respect to y, using the Fundamental Theorem of Calculus on the left:

$$\int_a^b g_y(x,y)dx = \frac{d}{dy} \int_a^b g(x,y)dx - 0$$

Richard Feynman

May 11, 1918 – February 15, 1988
Nobel Prize in Physics, 1965



"I used that one damn tool again and again."

" I caught on how to use that method, and I used that one damn tool again and again. [If] guys at MIT or Princeton had trouble doing a certain integral, [then] I come along and try differentiating under the integral sign, and often it worked. So I got a great reputation for doing integrals, only because my box of tools was different from everybody else's, and they had tried all their tools on it before giving the problem to me. (*Surely You're Joking, Mr. Feynman!*)

Richard Feynman's Integral Trick