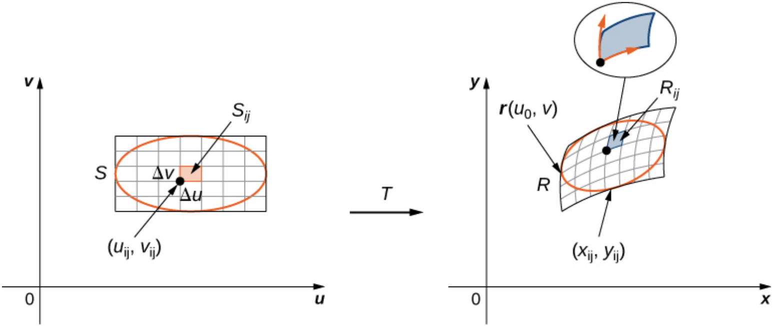
MATH 224: Vector Calculus



Class 24: April 10, 2026

Department of Mathematics and Statistics

Pre-registration Dessert Social

Wednesday, 4/15 | 3:30-4:30pm | Warner 105

Interested in taking some Math or Stat courses in **Fall 2026?** Currently taking a Math or Stats class? Need a study break?

Join the Math & Stats faculty over dessert to:

- Learn about Fall 2026 course offerings
- Get information about:
  - Major in Mathematics and/or the Applied Math Track
  - Major in Statistics
  - Minor in Mathematics
- Ask questions and receive advice about how Math and Stats fits into your Middlebury experience
- Be in community and hear from other students about Math and Stat courses

Anyone who is currently taking or wants to take a Math or Stats course is welcome! Even if you're graduating in May, we hope to see you at the dessert social!

Notes on Assignment 21

Assignment 22

Jacobi's Theorem on Change of Variable

Project

Project Teams

Announcements I

Comments/Questions on Exam 2?
Median Score: 89.5

Opportunity	Date	Weight
Exam 3	Monday, May 4	20%
Project	Friday, May 8	10%
Final Exam	Friday, May 15	30%

Announcements II

Review Improper Integrals:

$$\int_1^\infty \frac{1}{x^n}\, dx$$

This Week/Next Week:

Change of Variable

Leibniz Rule

Improper Integrals

Application to Probability

Leibniz Rule: Interchanging Differentiation and Integration

If g_y is continuous on $a \le x \le b, c \le y \le d$, then

$$\frac{d}{dy}\int_a^b g(x,y)dx = \int_a^b \frac{\partial}{\partial y} g(x,y)dx$$

$$\frac{d}{dy}\int_a^b g(x,y)dx = \int_a^b \frac{\partial}{\partial y} g(x,y)dx$$

Example Compute $f(x) = \int_0^1 \frac{u^x - 1}{\ln u} du$

By Leibniz:

$$f'(x) = \int_0^1 \frac{1}{\ln u}(u^x \ln u)du = \int_0^1 u^x du = \frac{u^{x+1}}{x+1}\bigg|_{u=0}^{u=1} = \frac{1}{x+1}$$

So $f(x) = \ln(x+1) + C$ for some constant C.

To Find C, evaluate at $x = 0$:

$f(0) = \int_0^1 \frac{u^0 - 1}{\ln u} du = \int_0^1 0 = 0$

But $f(0) = \ln(0+1) + C = \ln(1) + C = 0 + C = C$ so $C = 0$ and

$f(x) = \ln(x+1)$

Example: Find $f'(y)$ if $f(y) = \int_0^1 (y^2 + t^2)dt$

Method I: $f(y) = \int_0^1 (y^2 + t^2)dt = (y^2 t + \frac{t^3}{3})\Big|_{t=0}^{t=1} = y^2 + \frac{1}{3}$ so

$$f'(y) = 2y$$

Method II: (Leibniz) $f'(y) = \int_0^1 2y dt = 2yt\Big|_0^1 = 2y$

Proof of Leibniz Rule

To Show:

$$\frac{d}{dy}\int_a^b g(x,y)dx = \int_a^b \frac{\partial}{\partial y} g(x,y)dx$$

Let $f(y) = \int_a^b g(x,y)dx$ and Use Definition of Derivative

$$f'(y) = \lim_{h\to 0} \frac{f(y+h)-f(y)}{h}$$

$$\frac{f(y+h)-f(y)}{h} = \frac{\int_a^b g(x,y+h)dx - \int_a^b g(x,y)dx}{h} = \frac{\int_a^b (g(x,y+h)-g(x,y))dx}{h}$$

$$f'(y) = \lim_{h\to 0} \frac{f(y+h) - f(y)}{h} = \lim_{h\to 0} \frac{\int_a^b [g(x,y+h) - g(x,y)]\, dx}{h}$$

Interchange Limit and Integral:

$$= \int_a^b \left(lim_{h\to 0} \frac{[g(x,y+h) - g(x,y)]}{h} \right) dx$$

$$= \int_a^b \frac{\partial g}{\partial y}(x,y) dx$$

Alternate Proof of Leibniz Rule

(**Uses Iterated Integral**)

Begin with $\int_a^b g_y(x,y)dx$

Let $I = \int_c^y (\int_a^b g_y(x,y)dx)dy$

Switch Order of Integration: $I = \int_a^b \left(\int_c^y g_y(x,y)dy\right) dx$

$$I = \int_a^b g(x,y)\Big|_{y=c}^{y=y} dx = \int_a^b g(x,y) - g(x,c)dx$$

$$= \int_a^b g(x,y)dx - \int_a^b g(x,c)dx$$

The left term is a function of y and the second is a constant C

Alternate Proof of Leibniz Rule (Continued)

$$I = \int_c^y \left(\int_a^b g_y(x,y)dx \right) dy = \int_a^b g(x,y)dx - C$$

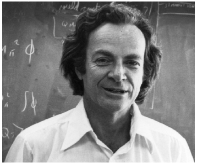
Now Take the Derivative of Each Side with Respect to y, using the Fundamental Theorem of Calculus on the left:

$$\int_a^b g_y(x,y)dx = \frac{d}{dy} \int_a^b g(x,y)dx - 0$$

Richard Feynman

May 11, 1918 – February 15, 1988

Nobel Prize in Physics, 1965



"I used that one damn tool again and again."

" I caught on how to use that method, and I used that one damn tool again and again. [If] guys at MIT or Princeton had trouble doing a certain integral, [then] I come along and try differentiating under the integral sign, and often it worked. So I got a great reputation for doing integrals, only because my box of tools was different from everybody else's, and they had tried all their tools on it before giving the problem to me. (*Surely You're Joking, Mr. Feynman!*)

Richard Feynman's Integral Trick

Change of Variable aka Method of Substitution

A common technique in the evaluation of integrals is to make a change of variable in the hopes of simplifying the problem of determining an antiderivatives

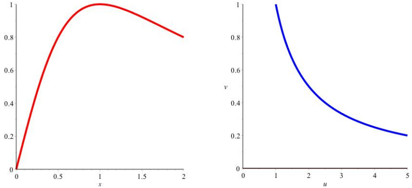
Example: Evaluate $\displaystyle\int_{x=0}^{x=2} \frac{2x}{1+x^2}\,dx$

$$\begin{array}{l|l} \text{Let } u = 1 + x^2 & x = 0 \to u = 1 + 0^2 = 1 \\ \text{The } du = 2x\,dx & x = 2 \to u = 1 + 2^2 = 5 \end{array}$$

$$\int_{x=0}^{x=2} \frac{2x}{1+x^2}\,dx = \int_{u=1}^{u=5} \frac{1}{u}\,du = \ln 5 - \ln 1 = \ln 5$$

$$\int_{x=0}^{x=2} \frac{2x}{1+x^2}\,dx = \int_{u=1}^{u=5} \frac{1}{u}\,du$$

Let's look at what is happening geometrically:



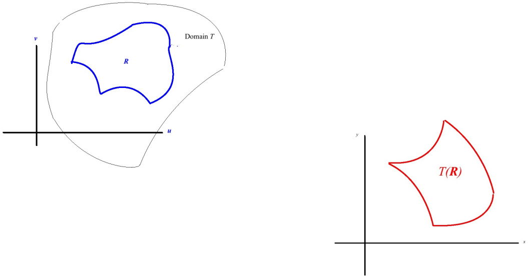
Not only does the function change, but also the region of integration.

The region of integration changes from an interval of length 2 to an interval of length 4.

The interval also moves to a new location.

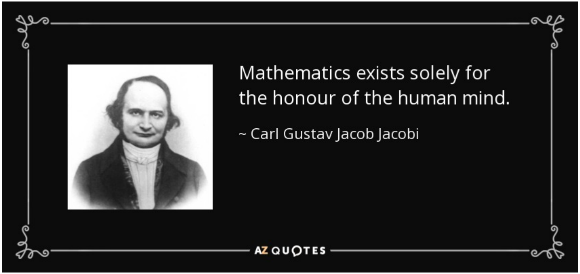
In computing mutltiple integrals, the corresponding change in the region may be more complicated.

By a **change of variable**, we will mean a vector function T from $\mathbb{R}^n$ to $\mathbb{R}^n$. It is convenient to use different letters to denote the spaces; e.g, $T : \mathbb{U}^n \to \mathbb{R}^n$



Carl Gustav Jacob Jacobi

December 10, 1804 – February 18, 1851



For further information see his Biography

Jacobi's Theorem

Let $\mathcal{R}$ be a set in $\mathbb{U}^n$ and $T(\mathcal{R})$ its image under T; that is,

$$T(\mathcal{R}) = \{T(\vec{u}) : \vec{u} \text{ is in } \mathcal{R}\}$$

Suppose $f : \mathbb{R}^n \to \mathbb{R}^1$ is a real-valued function.

Then, under suitable conditions,

$$\int_{T(\mathcal{R})} f(\vec{x}) dV_{\vec{x}} = \int_{\mathcal{R}} f(T(\vec{u}))|det T'(\vec{u})| dV_{\vec{u}}$$

- T is continuous differentiable
- Boundary of $\mathcal{R}$ is finitely many smooth curves
- T is one-to-one on interior of $\mathcal{R}$
- The Jacobian Determinant det T' is non zero on interior of $\mathcal{R}$.
- The function f is bounded and continuous on $T(\mathcal{R})$

$$\int_{T(\mathcal{R})} f(\vec{x}) dV_{\vec{x}} = \int_{\mathcal{R}} f(T(\vec{u})|det T'(\vec{u}))| dV_{\vec{u}}$$

In our example: $u = 1 + x^2$ so $x = \sqrt{u-1}$
Thus $T(u) = \sqrt{u-1} = (u-1)^{1/2}$ so
$T'(u) = \frac{1}{2}(u-1)^{-1/2} = \frac{1}{2\sqrt{u-1}}$
$\int_0^2 \frac{2x}{1+x^2} dx = \int_{T(\mathcal{R})} f(\vec{x}) dV_{\vec{x}} = \int_1^5 f(T(u)|det T'(u)| du$

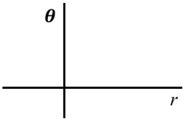
$$\text{Now } f(T(\vec{u}) = \frac{2T(u)}{1+(T(u))^2} = \frac{2\sqrt{u-1}}{1+u-1} = \frac{2\sqrt{u-1}}{u}$$

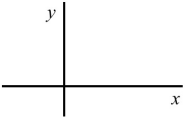
$$\det T'(u) = \left| \frac{1}{2\sqrt{u-1}} \right| = \frac{1}{2\sqrt{u-1}} \text{ so } f(T(\vec{u}) \det T'(u) = \frac{1}{u}$$

so $\int_0^2 \frac{2x}{1+x^2} dx = \int_1^5 \frac{2\sqrt{u-1}}{u} \frac{1}{2\sqrt{u-1}} du = \int_1^5 \frac{1}{u} du$

Example: **Polar Coordinate Change of Variable**

$\mathcal{U}^2 \qquad T \rightarrow \qquad \mathcal{R}^2$





$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$T(r,\theta) = (r\cos\theta, r\sin\theta)$$

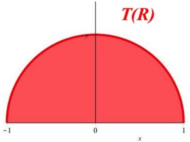
$$T' = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \text{ so } \det T' = r\cos^2\theta + r\sin^2\theta = r$$

Thus $\int_{T(R)} f(x,y)\,dx\,dy = \int_R f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$

$$\int_{T(R)} f(x,y)\, dx\, dy = \int_R f(r\cos\theta, r\sin\theta) r\, dr\, d\theta$$

$\underline{\text{Example}}$: $f(x,y) = x^2 + y^2$

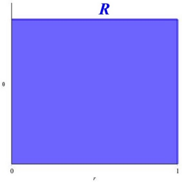
$$T(R) = \text{ Half Disk } = \{(x,y) : -1 \le x \le 1, 0 \le y \le \sqrt{1-x^2}\}$$



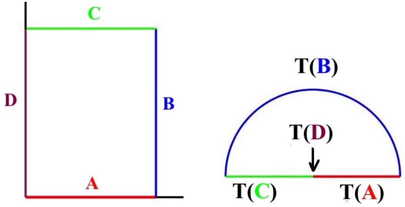
$$I = \int_{-1}^{1} \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx$$

Describe Region in Polar Coordinates: $0 \le r \le 1, 0 \le \theta \le \pi$

$$I = \int_{\theta=0}^{\pi} \int_{r=0}^{1} r^2\, r\, dr\, d\theta = \int_{\theta=0}^{\pi} \frac{r^4}{4}\bigg|_0^1 d\theta = \int_{\theta=0}^{\pi} \frac{1}{4} d\theta = \frac{\pi}{4}$$



Look At This Transformation More Closely



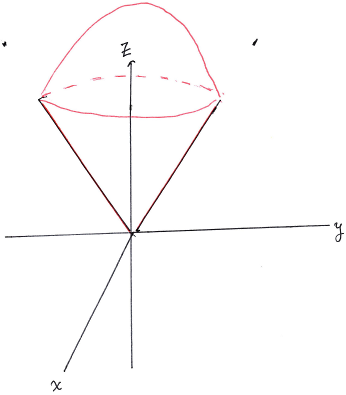
$A : 0 \leq r \leq 1, \theta = 0$
$x = r\cos\theta = r\cos 0 = r$
$y = r\sin\theta = r\sin 0 = 0$

$B : r = 1, 0 \leq \theta \leq \pi$
$x = r\cos\theta = \cos\theta$
$y = r\sin\theta = \sin\theta$

$C : 0 \leq r \leq 1, \theta = \pi$
$x = r\cos\theta = r\cos\pi = -r$
$y = r\sin\theta = r\sin\pi = 0$

$D : r = 0, 0 \leq \theta \leq \pi$
$x = r\cos\theta = 0$
$y = r\sin\theta = 0$

Problem: Evaluate $\iiint_C \sqrt{x^2+y^2+z^2}\,dV$ where C is the ice cream cone $\{(x,y,z) : x^2+y^2+z^2 \leq 1, x^2+y^2 \leq \frac{z^2}{3}, z \geq 0\}$



Example: **Spherical Coordinates**

$$x = r\sin\phi\cos\theta \qquad T : (r,\phi,\theta) \to (x,y,z)$$
$$y = r\sin\phi\sin\theta \qquad \det\ T' = r^2\sin\phi$$
$$z = r\cos\phi$$

Problem: Evaluate $\iiint_C \sqrt{x^2+y^2+z^2}\,dV$

where C is the ice cream cone

$$\{(x,y,z) : x^2+y^2+z^2 \le 1, x^2+y^2 \le \frac{z^2}{3}, z \ge 0\}$$

$z \ge 0$ implies $\phi \le \frac{\pi}{2}$

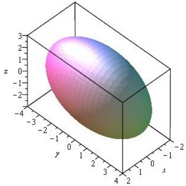
$x^2+y^2+z^2 \le 1$ implies $r \le 1$

$x^2+y^2 \le \frac{z^2}{3}$ implies $r^2\sin^2\phi \le \frac{r^2\cos^2\phi}{3}$

implies $\tan^2\phi \le \frac{1}{3}$ implies $\phi \le \frac{\pi}{6}$

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/6}\int_{r=0}^{1} \sqrt{r^2}\,r^2\sin\phi\,dr\,d\phi\,d\theta$$

Example: Evaluate $\iiint_D z^2 \, dV$ where D is the interior of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} = 1$



STEP 1: Let $u = \frac{x}{2}, v = \frac{y}{4}, w = \frac{z}{3}$.

Equation of the ellipsoid becomes $u^2 + v^2 + w^2 = 1$ (unit sphere)

So $x = 2u, y = 4v, z = 3w$ gives $T(u, v, w) = (2u, 4v, 3w)$ and

$$T' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} \text{ so } \det T' = 2 \times 4 \times 3 = 24$$

Thus $\iiint_D z^2 = \iiint (3w)^2 (24) \, du \, dv \, dw = 216 \iiint w^2 \, du \, dv \, dw$

STEP 2: Switch to Spherical Coordinates:

$$u = r\sin\phi\cos\theta, v = r\sin\phi\sin\theta, w = r\cos\phi$$

$$216\iiint w^2\,du\,dv\,dw = 216\iiint (r\cos\phi)^2 r^2\sin\phi\,dr\,d\phi\,d\theta$$

$$= 216\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{r=0}^{1} r^4\cos^2\phi\sin\phi\,dr\,d\phi\,d\theta$$

$$= (216)(2\pi)\int_{\phi=0}^{\pi}\int_{r=0}^{1} r^4\cos^2\phi\sin\phi\,dr\,d\phi$$

$$= (216)(2\pi)\frac{1}{5}\int_{\phi=0}^{\pi}\cos^2\phi\sin\phi\,d\phi$$

$$= \frac{(216)(2\pi)}{5}\left[-\frac{\cos^3\phi}{3}\right]_{\phi=0}^{\pi} = \frac{(216)(2\pi)}{5}\frac{2}{3} = \frac{288\pi}{5}$$