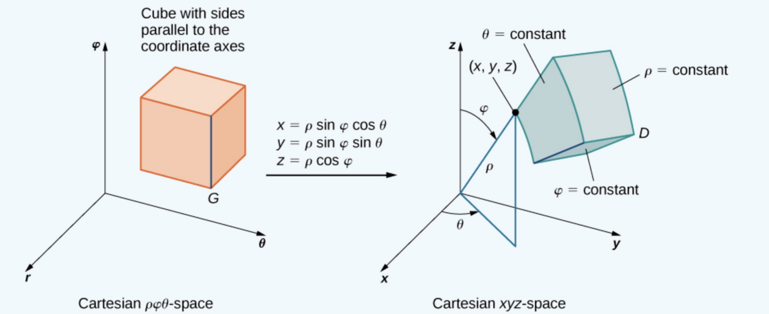
MATH 223: Vector Calculus



Class 25: Monday, April 13, 2026

Department of Mathematics and Statistics

Pre-registration Dessert Social

Wednesday, 4/15 | 3:30-4:30pm | Warner 105

Interested in taking some Math or Stat courses in **Fall 2026?** Currently taking a Math or Stats class? Need a study break?

Join the Math & Stats faculty over dessert to:

- Learn about Fall 2026 course offerings
- Get information about:
  - Major in Mathematics and/or the Applied Math Track
  - Major in Statistics
  - Minor in Mathematics
- Ask questions and receive advice about how Math and Stats fits into your Middlebury experience
- Be in community and hear from other students about Math and Stat courses

Anyone who is currently taking or wants to take a Math or Stats course is welcome! Even if you're graduating in May, we hope to see you at the dessert social!

Handouts

Notes on Assignment 22
Assignment 23
Improper Integrals and Probability Density Functions

Announcements

Today

Change of Variable

Improper Integrals

Application to Probability

Change of Variable aka Method of Substitution

A common technique in the evaluation of integrals is to make a change of variable in the hopes of simplifying the problem of determining an antiderivatives

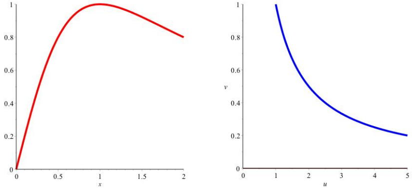
Example: Evaluate $\displaystyle\int_{x=0}^{x=2} \frac{2x}{1+x^2}dx$

$$\begin{array}{l|l} \text{Let } u = 1 + x^2 & x = 0 \to u = 1 + 0^2 = 1 \\ \text{The } du = 2xdx & x = 2 \to u = 1 + 2^2 = 5 \end{array}$$

$$\int_{x=0}^{x=2} \frac{2x}{1+x^2}dx = \int_{u=1}^{u=5} \frac{1}{u}du = \ln 5 - \ln 1 = \ln 5$$

$$\int_{x=0}^{x=2} \frac{2x}{1+x^2}\,dx = \int_{u=1}^{u=5} \frac{1}{u}\,du$$

Let's look at what is happening geometrically:



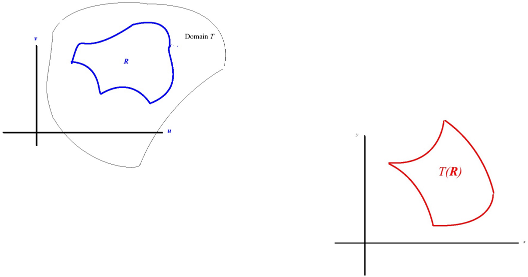
Not only does the function change, but also the region of integration.

The region of integration changes from an interval of length 2 to an interval of length 4.

The interval also moves to a new location.

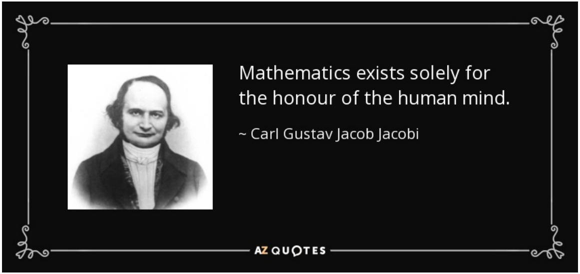
In computing mutltiple integrals, the corresponding change in the region may be more complicated.

By a **change of variable**, we will mean a vector function T from $\mathbb{R}^n$ to $\mathbb{R}^n$. It is convenient to use different letters to denote the spaces; e.g, $T : \mathbb{U}^n \to \mathbb{R}^n$



Carl Gustav Jacob Jacobi

December 10, 1804 – February 18, 1851



For further information see his Biography

Jacobi's Theorem

Let $\mathcal{R}$ be a set in $\mathbb{U}^n$ and $T(\mathcal{R})$ its image under T; that is,

$$T(\mathcal{R}) = \{T(\vec{u}) : \vec{u} \text{ is in } \mathcal{R}\}$$

Suppose $f : \mathbb{R}^n \to \mathbb{R}^1$ is a real-valued function.

Then, under suitable conditions,

$$\int_{T(\mathcal{R})} f(\vec{x}) dV_{\vec{x}} = \int_{\mathcal{R}} f(T(\vec{u})) |det T'(\vec{u})| dV_{\vec{u}}$$

- T is continuous differentiable
- Boundary of $\mathcal{R}$ is finitely many smooth curves
- T is one-to-one on interior of $\mathcal{R}$
- The Jacobian Determinant det T' is non zero on interior of $\mathcal{R}$.
- The function f is bounded and continuous on $T(\mathcal{R})$

$$\int_{T(\mathcal{R})} f(\vec{x}) dV_{\vec{x}} = \int_{\mathcal{R}} f(T(\vec{u})|det T'(\vec{u}))| dV_{\vec{u}}$$

In our example: $u = 1 + x^2$ so $x = \sqrt{u-1}$
Thus $T(u) = \sqrt{u-1} = (u-1)^{1/2}$ so
$T'(u) = \frac{1}{2}(u-1)^{-1/2} = \frac{1}{2\sqrt{u-1}}$
$\int_0^2 \frac{2x}{1+x^2} dx = \int_{T(\mathcal{R})} f(\vec{x}) dV_{\vec{x}} = \int_1^5 f(T(u)|det T'(u)| du$

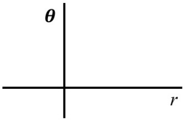
$$\text{Now } f(T(\vec{u}) = \frac{2T(u)}{1+(T(u))^2} = \frac{2\sqrt{u-1}}{1+u-1} = \frac{2\sqrt{u-1}}{u}$$

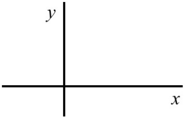
$$\det T'(u) = \left| \frac{1}{2\sqrt{u-1}} \right| = \frac{1}{2\sqrt{u-1}} \text{ so } f(T(\vec{u}) \det T'(u) = \frac{1}{u}$$

so $\int_0^2 \frac{2x}{1+x^2} dx = \int_1^5 \frac{2\sqrt{u-1}}{u} \frac{1}{2\sqrt{u-1}} du = \int_1^5 \frac{1}{u} du$

Example: Polar Coordinate Change of Variable

$\mathcal{U}^2 \qquad T \rightarrow \qquad \mathcal{R}^2$





$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$T(r,\theta) = (r\cos\theta, r\sin\theta)$$

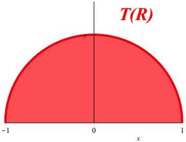
$$T' = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \text{ so } \det T' = r\cos^2\theta + r\sin^2\theta = r$$

Thus $\int_{T(R)} f(x,y)\,dx\,dy = \int_R f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$

$$\int_{T(R)} f(x,y)\,dx\,dy = \int_R f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

Example: $f(x,y) = x^2 + y^2$

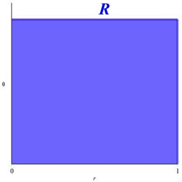
$T(R) =$ Half Disk $= \{(x,y) : -1 \le x \le 1, 0 \le y \le \sqrt{1-x^2}\}$



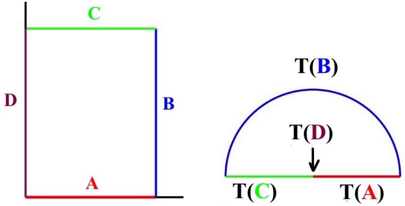
$$I = \int_{-1}^{1} \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\,dy\,dx$$

Describe Region in Polar Coordinates: $0 \le r \le 1, 0 \le \theta \le \pi$

$$I = \int_{\theta=0}^{\pi} \int_{r=0}^{1} r^2\, r\,dr\,d\theta = \int_{\theta=0}^{\pi} \left.\frac{r^4}{4}\right|_0^1 d\theta = \int_{\theta=0}^{\pi} \frac{1}{4} d\theta = \frac{\pi}{4}$$

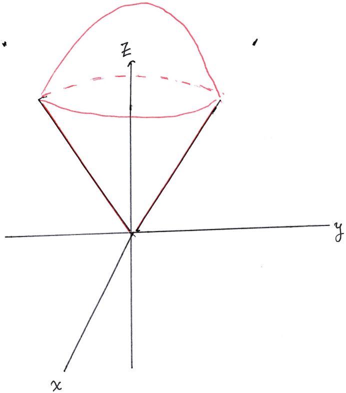


Look At This Transformation More Closely



$A: 0 \leq r \leq 1, \theta = 0$ $x = r\cos\theta = r\cos 0 = r$ $y = r\sin\theta = r\sin 0 = 0$	$B: r = 1, 0 \leq \theta \leq \pi$ $x = r\cos\theta = \cos\theta$ $y = r\sin\theta = \sin\theta$
$C: 0 \leq r \leq 1, \theta = \pi$ $x = r\cos\theta = r\cos\pi = -r$ $y = r\sin\theta = r\sin\pi = 0$	$D: r = 0, 0 \leq \theta \leq \pi$ $x = r\cos\theta = 0$ $y = r\sin\theta = 0$

Problem: Evaluate $\iiint_C \sqrt{x^2+y^2+z^2}\,dV$
where C is the ice cream cone
$\{(x,y,z) : x^2+y^2+z^2 \le 1, x^2+y^2 \le \frac{z^2}{3}, z \ge 0\}$



Example: **Spherical Coordinates**

$$x = r\sin\phi\cos\theta \qquad T : (r,\phi,\theta) \to (x,y,z)$$
$$y = r\sin\phi\sin\theta \qquad \det\ T' = r^2\sin\phi$$
$$z = r\cos\phi$$

Problem: Evaluate $\iiint_C \sqrt{x^2+y^2+z^2}\,dV$

where C is the ice cream cone

$$\{(x,y,z) : x^2+y^2+z^2 \le 1, x^2+y^2 \le \frac{z^2}{3}, z \ge 0\}$$

$z \ge 0$ implies $\phi \le \frac{\pi}{2}$

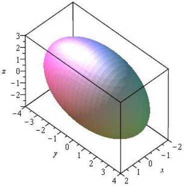
$x^2+y^2+z^2 \le 1$ implies $r \le 1$

$x^2+y^2 \le \frac{z^2}{3}$ implies $r^2\sin^2\phi \le \frac{r^2\cos^2\phi}{3}$

implies $\tan^2\phi \le \frac{1}{3}$ implies $\phi \le \frac{\pi}{6}$

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/6}\int_{r=0}^{1} \sqrt{r^2}\,r^2\sin\phi\,dr\,d\phi\,d\theta$$

Example: Evaluate $\iiint_D z^2 \, dV$ where D is the interior of the ellipsoid $\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} = 1$



STEP 1: Let $u = \frac{x}{2}, v = \frac{y}{4}, w = \frac{z}{3}$.

Equation of the ellipsoid becomes $u^2 + v^2 + w^2 = 1$ (unit sphere)

So $x = 2u, y = 4v, z = 3w$ gives $T(u, v, w) = (2u, 4v, 3w)$ and

$$T' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} \text{ so } \det T' = 2 \times 4 \times 3 = 24$$

Thus $\iiint_D z^2 = \iiint (3w)^2 (24) \, du \, dv \, dw = 216 \iiint w^2 \, du \, dv \, dw$

STEP 2: Switch to Spherical Coordinates:

$$u = r\sin\phi\cos\theta, v = r\sin\phi\sin\theta, w = r\cos\phi$$

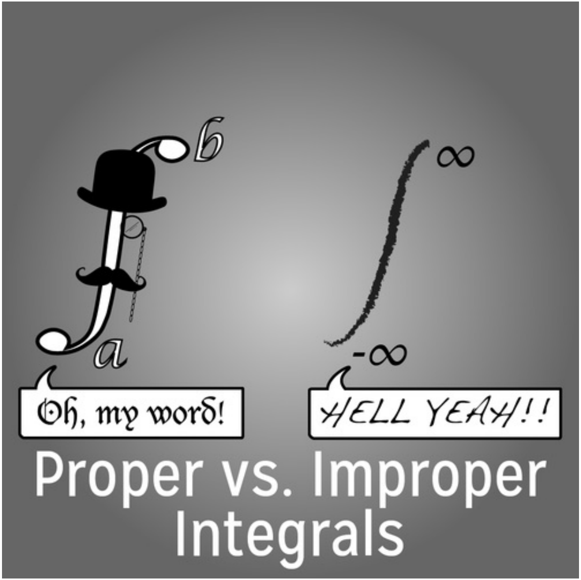
$$216\iiint w^2\, du\, dv\, dw = 216\iiint (r\cos\phi)^2 r^2 \sin\phi\, dr\, d\phi\, d\theta$$

$$= 216\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{r=0}^{1} r^4\cos^2\phi\sin\phi\, dr\, d\phi\, d\theta$$

$$= (216)(2\pi)\int_{\phi=0}^{\pi}\int_{r=0}^{1} r^4\cos^2\phi\sin\phi\, dr\, d\phi$$

$$= (216)(2\pi)\frac{1}{5}\int_{\phi=0}^{\pi}\cos^2\phi\sin\phi\, d\phi$$

$$= \frac{(216)(2\pi)}{5}\left[-\frac{\cos^3\phi}{3}\right]_{\phi=0}^{\pi} = \frac{(216)(2\pi)}{5}\frac{2}{3} = \frac{288\pi}{5}$$


b
a
Oh, my word!
∞
-∞
HELL YEAH!!
Proper vs. Improper
Integrals

Improper Integrals

Setting $\int_{\mathcal{B}} f\, dV$ where $\mathcal{B}$ is a subset of $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^1$

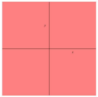
Two Types:

(I): $\mathcal{B}$ is unbounded

(II) $\mathcal{B}$ is bounded but f is unbounded

Type I Examples

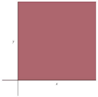
$\mathcal{B} = \mathbb{R}^2$



$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dy\, dx$

$\int_{r=0}^{\infty} \int_{\theta=0}^{2\pi} f^*(r, \theta) r\, d\theta\, dr$

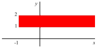
$\mathcal{B} =$ First Quadrant



$\int_{0}^{\infty} \int_{0}^{\infty} f(x, y)\, dy\, dx$

$\int_{r=0}^{\infty} \int_{\theta=0}^{\pi/2} f^*(r, \theta) r\, d\theta\, dr$

$\mathcal{B}$ is infinite strip



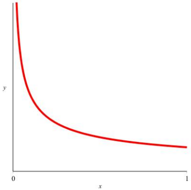
. $\int_{-1}^{\infty} \int_{1}^{2} f(x, y)\, dy\, dx$

$$\int_{-1}^{\infty} \int_{1}^{2} f(x, y)\, dy\, dx = \lim_{b\to\infty} \int_{-1}^{b} \int_{1}^{2} f(x, y)\, dy\, dx$$

Type II Examples

Classic Case

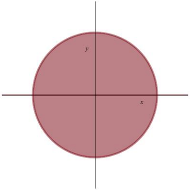
$$I = \int_0^1 \frac{1}{\sqrt{x}}\, dx$$



$$I = \lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\, dx = \lim_{a\to 0^+} \left[2\sqrt{x}\right]_a^1 = \lim_{a\to 0^+} \left[2 - 2\sqrt{a}\right] = 2$$

Type II Examples

In $\mathbb{R}^2, f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$ on unit disk



Type II Examples

In Polar Coordinates:

$$\int_0^1 \int_0^{2\pi} \frac{1}{r}\, r\, d\theta\, dr = \lim_{a \to 0^+} \int_a^1 \int_0^{2\pi} d\theta\, dr = \lim_{a \to 0^+} \int_a^1 2\pi dr$$

$$= \lim_{a \to 0^+} (2\pi - 2\pi a) = 2\pi$$


ONE DOES NOT SIMPLY
WRITE AN IMPROPER INTEGRAL
WITHOUT THE LIMIT

Improper Integrals

Let $\{B_\delta\}$ be a family of bounded sets B_δ that expands to cover all of the set B. We say $\int_B f(\mathbf{x})dV$ is defined as an **improper integral** if the limit

$$\int_B f(\mathbf{x})dV = \lim \int_{B_\delta} f(\mathbf{x})\,dV$$

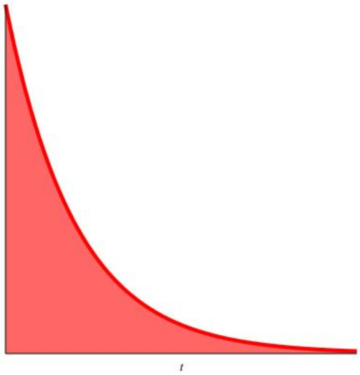
is finite and independent of the family $\{B_\delta\}$ used to define it. If the limit exists (as a finite number), we say that the improper integral **converges** to that value. If the limit fails to exist, we say the improper integral **diverges.**

An Important Example:

Exponential Probability Density Function

$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty}\left[-e^{-x}\Big|_{x=0}^{b}\right]$$

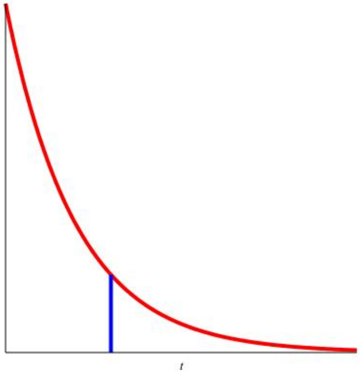
$$= \lim_{b\to\infty}\left[-e^{-b}-(-e^0)\right] = \lim_{b\to\infty}\left[1-\frac{1}{e^b}\right] = 1$$



Exponential Probability Density Function

Probability(Light Bulb Burns Out in $\leq x$ months) $=$

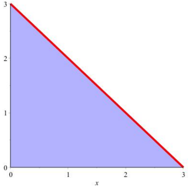
$$\int_0^x e^{-t}\,dt = 1 - e^{-x}$$



x	$\int_0^x e^{-t}dt$	Prob(Bulb Lasts More than x months)
1	.632	.368
2	.865	.135
3	.950	.050
4	.982	.018

Suppose You Buy 2 Light Bulbs

What Is The Probability They Will Provide At Least 3 Months of Service?



$$\text{Prob}(x + y > 3) = 1 \text{ - } \text{Prob}(x + y \leq 3)$$

$$= 1 - \int_{x=0}^{3} \int_{y=0}^{3-x} e^{-x} e^{-y} \, dy \, dx$$

Evaluate $1 - \int_{x=0}^{3} \int_{y=0}^{3-x} e^{-x} e^{-y} \, dy \, dx$

$$= 1 - \int_0^3 e^{-x} \left[-e^{-y} \Big|_{y=0}^{3-x} \right] dx$$

$$= 1 - \int_0^3 e^{-x} \left[-e^{3-x} + 1 \right] dx$$

$$= 1 - \int_0^3 (e^{-x} - e^{-3}) \, dx$$

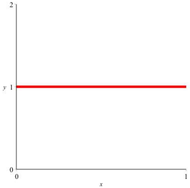
$$= 1 - \left[-e^{-x} - e^{-3} x \right]_{x=0}^{3}$$

$$= 1 - \left[-e^{-3} - 3e^{-3} + 1 + 0 \right] = 1 - \left[1 - \frac{4}{e^3} \right] = \frac{4}{e^3} \approx .199$$

Probability Density Function

A real-valued function p such that $p(\vec{x}) \geq 0$ for all $\vec{x}$ and $\int_S p = 1$ where S is the set of all possibilities.

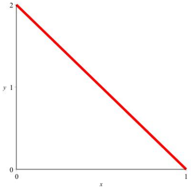
Example 1 Uniform Density: $p(x) = 1$ on [0,1]



$$\int_S p = \int_0^1 1 = x\Big|_0^1 = 1$$

Example 2: $p(x) = 2 - 2x$ on [0,1]

More likely to choose small numbers than larger numbers



Problem: Find the probability of picking a number less than 1/2.

$$\int_0^{1/2}(2-2x)\,dx = (2x - x^2)\Big|_0^{1/2} = (1 - \tfrac{1}{4}) - (0-0) = \tfrac{3}{4}$$

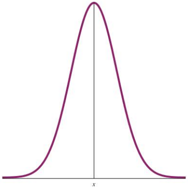
A probability density function on a set S in $\mathbb{R}^n$ is a continuous non-negative real-valued function $p : S \to \mathbb{R}^1$ such that

$$\int_S p dV = 1$$

If an experiment is performed where S is the set of all possible outcomes, then the probability that the outcome lies in a particular subset T is $\int_T p(\vec{x})\, dV$.

Example: **The Bell Curve**: The most important curve in statistics

Start with $y = e^{-\frac{x^2}{2}}$



Then $y' = -xe^{-\frac{x^2}{2}}$ and $y'' = (x^2 - 1)e^{-\frac{x^2}{2}}$

Point of inflection at $(1, \frac{1}{\sqrt{e}}) = (1, .606)$

Need to find $A = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \, dx$

Impossible to find antiderivative of $e^{-\frac{x^2}{2}}$

Need to find $A = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx$

$$A^2 = \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{-x^2-y^2}{2}}\, dy\, dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}}\, dy\, dx$$

Switch To Polar Coordinates: $A^2 = \int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} e^{-\frac{r^2}{2}}\, r\, d\theta\, dr$

$$A^2 = 2\pi \int_{r=0}^{\infty} re^{-\frac{r^2}{2}}\, dr = 2\pi \lim_{b\to\infty} \int_{r=0}^{b} re^{-\frac{r^2}{2}}\, dr$$

$$= 2\pi \lim_{b\to\infty} \left[-e^{-\frac{r^2}{2}}\right]_0^b = 2\pi \lim_{b\to\infty} \left[-\frac{1}{e^{b^2/2}} + \frac{1}{e^0}\right] = 2\pi \times 1 = 2\pi$$

Thus $A^2 = 2\pi$ so $A = \sqrt{2\pi}$

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}$$

To get a probability density, let $p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$
This density is called the **Standard Normal Density**

Example: Suppose two numbers b and c are chosen at random between 0 and 1.

What is the probability that the quadratic equation $x^2 + bx + c = 0$ has a real root?

Solution: Choosing b and c is equivalent to choosing a point (b, c) from the unit square S with $p(\vec{x}) = 1$ (**Uniform Density**)

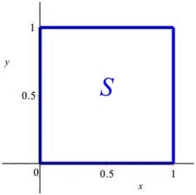
Then $\int_S p(\vec{x}) = \int_S 1 = area(S) = 1$.

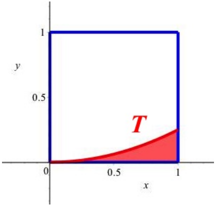
Now $x^2 + bx + c = 0$ has solution $x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$

For real root, need $b^2 - 4c \geq 0$ or $c \leq \frac{b^2}{4}$

Let $T = \{(b, c) : c \leq \frac{b^2}{4}\}$

$$\int_T p(\vec{x}) = \int_{x=0}^{1} \int_{y=0}^{x^2/4} 1 \, dy \, dx = \int_{x=0}^{1} \frac{x^2}{4} \, dx = \left. \frac{x^3}{12} \right|_0^1 = \frac{1}{12}$$





General Exponential Probability Distribution

$$p(x) = \lambda e^{-\lambda x} \text{ for } x \geq 0, \lambda > 0$$

Easy to Show:

$$\int_0^\infty \lambda e^{-\lambda x}\, dx = 1 \text{ so it is a probability distribution}$$

$$\text{Mean } \int_0^\infty \lambda x e^{-\lambda x}\, dx = \frac{1}{\lambda}$$

$$\text{Prob(Bulb life} \geq 3) = 1 - \int_3^\infty \lambda e^{-\lambda x} dx = 1 + e^{-\lambda x}\Big|_3^\infty = 1 - e^{-3\lambda}$$

$$\text{Prob(2 lights have life} \geq 3) = e^{-3\lambda}(1 + 3\lambda)$$

More than b hours: $e^{-3b\lambda}(1 + b\lambda)$