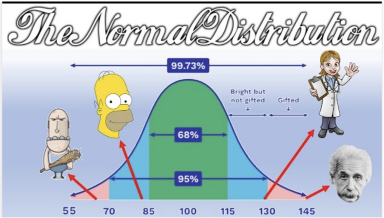
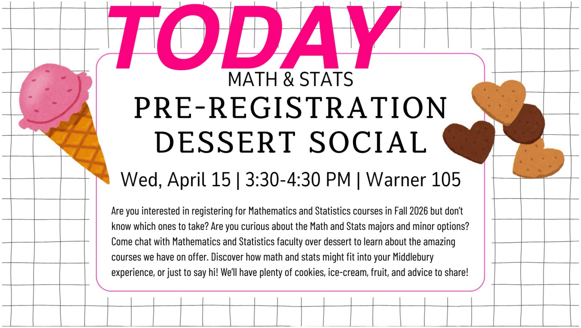
MATH 224: Vector Calculus



Class 25: Wednesday, April 15, 2026


TODAY
MATH & STATS
PRE-REGISTRATION
DESSERT SOCIAL
Wed, April 15 | 3:30-4:30 PM | Warner 105
Are you interested in registering for Mathematics and Statistics courses in Fall 2026 but don't know which ones to take? Are you curious about the Math and Stats majors and minor options? Come chat with Mathematics and Statistics faculty over dessert to learn about the amazing courses we have on offer. Discover how math and stats might fit into your Middlebury experience, or just to say hi! We'll have plenty of cookies, ice-cream, fruit, and advice to share!

Handouts

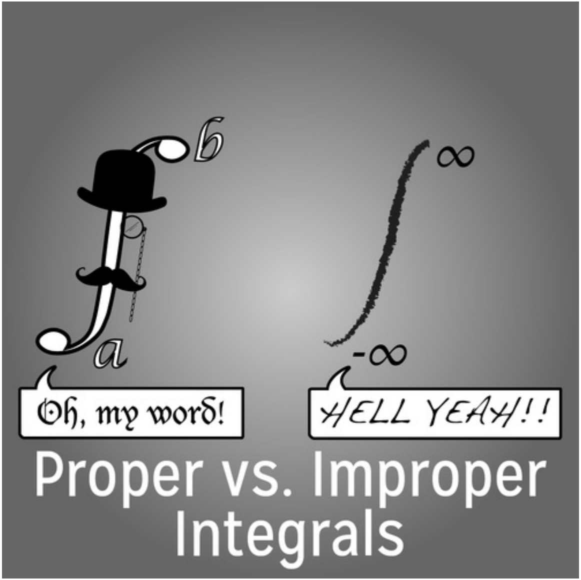
Notes on Assignment 23
Assignment 24
Improper Integrals and Probability Density Functions

Announcements

Today

Improper Integrals

Application to Probability


b
a
∞
-∞
Oh, my word!
HELL YEAH!!
Proper vs. Improper
Integrals

Improper Integrals

Setting $\int_{\mathcal{B}} f \, dV$ where $\mathcal{B}$ is a subset of $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^1$

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Two Types:

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Two Types:

(I): $\mathcal{B}$ is unbounded

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(I): $\mathcal{B}$ is unbounded

(II) $\mathcal{B}$ is bounded but f is unbounded

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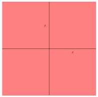
Type I Examples

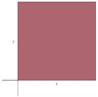
$\mathcal{B} = \mathbb{R}^2$



$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dy\, dx$

$\int_{r=0}^{\infty} \int_{\theta=0}^{2\pi} f^*(r,\theta) r\, d\theta\, dr$

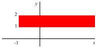
$\mathcal{B} =$ First Quadrant



$\int_{0}^{\infty} \int_{0}^{\infty} f(x,y)\, dy\, dx$

$\int_{r=0}^{\infty} \int_{\theta=0}^{\pi/2} f^*(r,\theta) r\, d\theta\, dr$

$\mathcal{B}$ is infinite strip



. $\int_{-1}^{\infty} \int_{1}^{2} f(x,y)\, dy\, dx$

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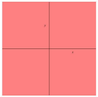

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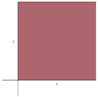

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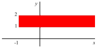


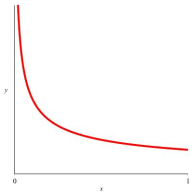
. $\int_{-1}^{\infty} \int_{1}^{2} f(x, y)\, dy\, dx$

$$\int_{-1}^{\infty} \int_{1}^{2} f(x, y)\, dy\, dx = \lim_{b \to \infty} \int_{-1}^{b} \int_{1}^{2} f(x, y)\, dy\, dx$$

Type II Examples

Classic Case

$$I = \int_0^1 \frac{1}{\sqrt{x}}\, dx$$



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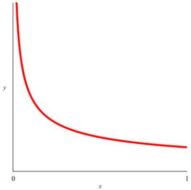
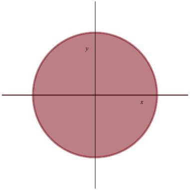


$$I = \lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} \left[2\sqrt{x}\right]_a^1 = \lim_{a\to 0^+} \left[2 - 2\sqrt{a}\right] = 2$$

Type II Examples

In $\mathbb{R}^2, f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$ on unit disk



Type II Examples

In Polar Coordinates:

$$\int_0^1 \int_0^{2\pi} \frac{1}{r}\, r\, d\theta\, dr = \lim_{a\to 0^+} \int_a^1 \int_0^{2\pi} d\theta\, dr = \lim_{a\to 0^+} \int_a^1 2\pi dr$$

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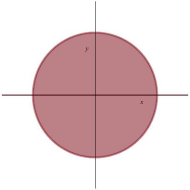

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$$= \lim_{a \to 0^+} (2\pi - 2\pi a) = 2\pi$$


ONE DOES NOT SIMPLY
WRITE AN IMPROPER INTEGRAL
WITHOUT THE LIMIT
generator.net

Improper Integrals

Let $\{B_\delta\}$ be a family of bounded sets B_δ that expands to cover all of the set B. We say $\int_B f(\mathbf{x})dV$ is defined as an **improper integral** if the limit $\int_B f(\mathbf{x})dV = \lim \int_{B_\delta} f(\mathbf{x})\,dV$ is finite and independent of the family $\{B_\delta\}$ used to define it. If the limit exists (as a finite number), we say that the improper integral **converges** to that value. If the limit fails to exist, we say the improper integral **diverges.**

An Important Example:

Exponential Probability Density Function

$$\int_0^{\infty} e^{-x}\, dx = \lim_{b \to \infty} \int_0^{b} e^{-x}\, dx = \lim_{b \to \infty} \left[-e^{-x} \Big|_{x=0}^{b} \right]$$

An Important Example:

Exponential Probability Density Function

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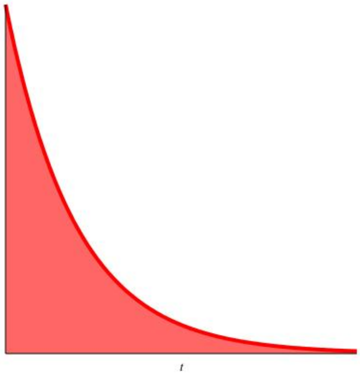
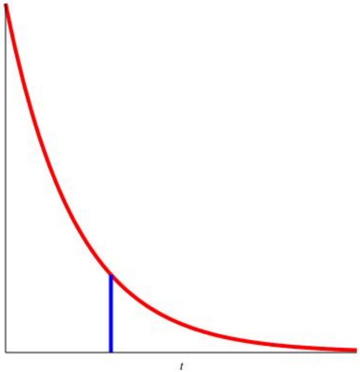


Exponential Probability Density Function

Probability(Light Bulb Burns Out in $\leq x$ months) $=$ $\int_0^x e^{-t}\,dt = 1 - e^{-x}$



x	$\int_0^x e^{-t}dt$	Prob(Bulb Lasts More than x months)
1	.632	.368
2	.865	.135
3	.950	.050
4	.982	.018

Suppose You Buy 2 Light Bulbs

What Is The Probability They Will Provide At Least 3 Months of Service?

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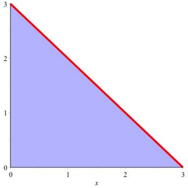

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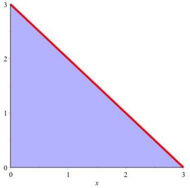

$$\text{Prob}(x + y > 3) = 1 \text{ - } \text{Prob}(x + y \leq 3)$$

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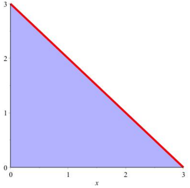


$$\text{Prob}(x + y > 3) = 1 \text{ - } \text{Prob}(x + y \leq 3)$$

$$= 1 - \int_{x=0}^{3} \int_{y=0}^{3-x} e^{-x} e^{-y} \, dy \, dx$$

$$\text{Evaluate } 1 - \int_{x=0}^{3} \int_{y=0}^{3-x} e^{-x} e^{-y} \, dy \, dx$$

$$= 1 - \int_0^3 e^{-x} \left[-e^{-y} \Big|_{y=0}^{3-x} \right] dx$$

$$= 1 - \int_0^3 e^{-x} \left[-e^{3-x} + 1 \right] dx$$

$$= 1 - \int_0^3 (e^{-x} - e^{-3}) \, dx$$

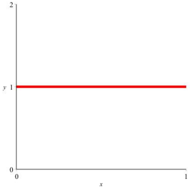
$$= 1 - \left[-e^{-x} - e^{-3} x \right]_{x=0}^{3}$$

$$= 1 - \left[-e^{-3} - 3e^{-3} + 1 + 0 \right] = 1 - \left[1 - \frac{4}{e^3} \right] = \frac{4}{e^3} \approx .199$$

Probability Density Function

A real-valued function p such that $p(\vec{x}) \geq 0$ for all $\vec{x}$ and $\int_S p = 1$ where S is the set of all possibilities.

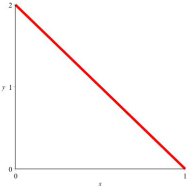
Example 1 Uniform Density: $p(x) = 1$ on [0,1]



$$\int_S p = \int_0^1 1 = x\Big|_0^1 = 1$$

Example 2: $p(x) = 2 - 2x$ on [0,1]

More likely to choose small numbers than larger numbers



Problem: Find the probability of picking a number less than $1/2$.

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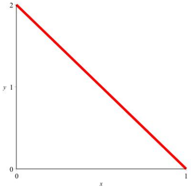

Problem: Find the probability of picking a number less than 1/2.

$$\int_0^{1/2} (2 - 2x)\, dx = (2x - x^2)\Big|_0^{1/2} = (1 - \tfrac{1}{4}) - (0 - 0) = \tfrac{3}{4}$$

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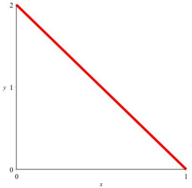

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A probability density function on a set S in $\mathbb{R}^n$ is a continuous non-negative real-valued function $p : S \to \mathbb{R}^1$ such that

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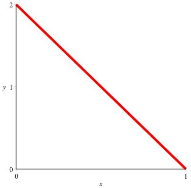

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A probability density function on a set S in $\mathbb{R}^n$ is a continuous non-negative real-valued function $p : S \to \mathbb{R}^1$ such that

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If an experiment is performed where S is the set of all possible outcomes, then the probability that the outcome lies in a particular subset T is $\int_T p(\vec{x})\,dV$.

Example: **The Bell Curve**: The most important curve in statistics

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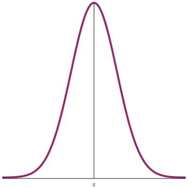
Start with $y = e^{-\frac{x^2}{2}}$



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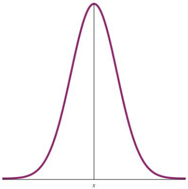

Then $y' = -xe^{-\frac{x^2}{2}}$ and $y'' = (x^2 - 1)e^{-\frac{x^2}{2}}$

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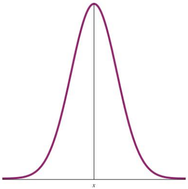

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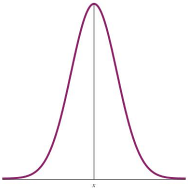

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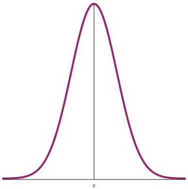

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Impossible to find antiderivative of $e^{-\frac{x^2}{2}}$

Need to find $A = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx$

Need to find $A = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx$

$$A^2 = \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{-x^2-y^2}{2}}\, dy\, dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}}\, dy\, dx$$

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Switch To Polar Coordinates: $A^2 = \int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} e^{-\frac{r^2}{2}}\, r\, d\theta\, dr$

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$$A^2 = 2\pi \int_{r=0}^{\infty} re^{-\frac{r^2}{2}}\, dr = 2\pi \lim_{b \to \infty} \int_{r=0}^{b} re^{-\frac{r^2}{2}}\, dr$$

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$$= 2\pi \lim_{b\to\infty} \left[-e^{-\frac{r^2}{2}}\right]_0^b = 2\pi \lim_{b\to\infty}\left[-\frac{1}{e^{b^2/2}} + \frac{1}{e^0}\right] = 2\pi \times 1 = 2\pi$$

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Thus $A^2 = 2\pi$ so $A = \sqrt{2\pi}$

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \, dx = \sqrt{2\pi}$$

To get a probability density, let $p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$
This density is called the **Standard Normal Density**

Example: Suppose two numbers b and c are chosen at random between 0 and 1.

What is the probability that the quadratic equation $x^2 + bx + c = 0$ has a real root?

Solution: Choosing b and c is equivalent to choosing a point (b, c) from the unit square S with $p(\vec{x}) = 1$ (**Uniform Density**)

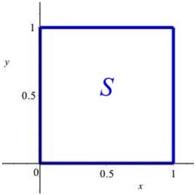
Then $\int_S p(\vec{x}) = \int_S 1 = area(S) = 1.$

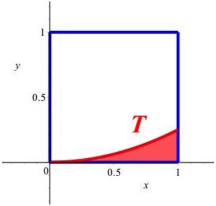
Now $x^2 + bx + c = 0$ has solution $x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$

For real root, need $b^2 - 4c \geq 0$ or $c \leq \frac{b^2}{4}$

Let $T = \{(b, c) : c \leq \frac{b^2}{4}\}$

$$\int_T p(\vec{x}) = \int_{x=0}^{1} \int_{y=0}^{x^2/4} 1 \, dy \, dx = \int_{x=0}^{1} \frac{x^2}{4} \, dx = \left. \frac{x^3}{12} \right|_0^1 = \frac{1}{12}$$





General Exponential Probability Distribution

$$p(x) = \lambda e^{-\lambda x} \text{ for } x \geq 0, \lambda > 0$$

Easy to Show:

$$\int_0^\infty \lambda e^{-\lambda x}\, dx = 1 \text{ so it is a probability distribution}$$

$$\text{Mean } \int_0^\infty \lambda x e^{-\lambda x}\, dx = \frac{1}{\lambda}$$

$$\text{Prob(Bulb life} \geq 3) = 1 - \int_3^\infty \lambda e^{-\lambda x} dx = 1 + e^{-\lambda x}\Big|_3^\infty = 1 - e^{-3\lambda}$$

$$\text{Prob(2 lights have life} \geq 3) = e^{-3\lambda}(1 + 3\lambda)$$

$$\text{More than } b \text{ hours: } e^{-3b\lambda}(1 + b\lambda)$$

What's Next?
Calculus Along Curves

- Work
- Vector Fields and Line Integrals
- Arc Length and Weighted Curves
- Curvature and Normals
- Flow Lines and Differential Equations