# MATH 224: Vector Calculus


Class 27: Monday April 20, 2026

Notes on Assignment 24

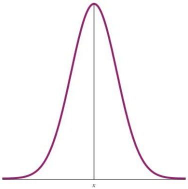
Assignment 25

INTEGRALS AND DERIVATIVES ON CURVES

Example: **The Bell Curve**: The most important curve in statistics

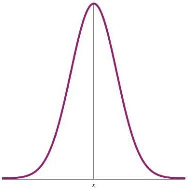
Example: **The Bell Curve**: The most important curve in statistics

Start with $y = e^{-\frac{x^2}{2}}$



Example: **The Bell Curve**: The most important curve in statistics

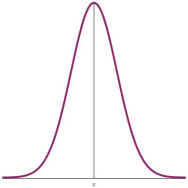
Start with $y = e^{-\frac{x^2}{2}}$




Then $y' = -xe^{-\frac{x^2}{2}}$ and $y'' = (x^2 - 1)e^{-\frac{x^2}{2}}$

Example: **The Bell Curve**: The most important curve in statistics

Start with $y = e^{-\frac{x^2}{2}}$

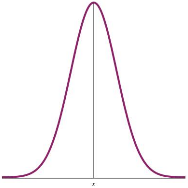



Then $y' = -xe^{-\frac{x^2}{2}}$ and $y'' = (x^2 - 1)e^{-\frac{x^2}{2}}$

Point of inflection at $(1, \frac{1}{\sqrt{e}}) = (1, .606)$

Example: **The Bell Curve**: The most important curve in statistics

Start with $y = e^{-\frac{x^2}{2}}$



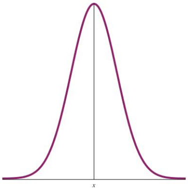
Then $y' = -xe^{-\frac{x^2}{2}}$ and $y'' = (x^2 - 1)e^{-\frac{x^2}{2}}$

Point of inflection at $(1, \frac{1}{\sqrt{e}}) = (1, .606)$

Need to find $A = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx$

Example: **The Bell Curve**: The most important curve in statistics

Start with $y = e^{-\frac{x^2}{2}}$



Then $y' = -xe^{-\frac{x^2}{2}}$ and $y'' = (x^2 - 1)e^{-\frac{x^2}{2}}$

Point of inflection at $(1, \frac{1}{\sqrt{e}}) = (1, .606)$

Need to find $A = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx$

Impossible to find antiderivative of $e^{-\frac{x^2}{2}}$

Need to find $A = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx$

Need to find $A = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx$

$$A^2 = \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{-x^2-y^2}{2}}\, dy\, dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}}\, dy\, dx$$

Need to find $A = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx$

$$A^2 = \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{-x^2-y^2}{2}}\, dy\, dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}}\, dy\, dx$$

**Switch To Polar Coordinates:** $A^2 = \displaystyle\int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} e^{-\frac{r^2}{2}}\, r\, d\theta\, dr$

Need to find $A = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx$

$$A^2 = \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{-x^2-y^2}{2}}\,dy\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}}\,dy\,dx$$

**Switch To Polar Coordinates:** $A^2 = \displaystyle\int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} e^{-\frac{r^2}{2}}\,r\,d\theta\,dr$

$$A^2 = 2\pi\int_{r=0}^{\infty} re^{-\frac{r^2}{2}}\,dr = 2\pi\lim_{b\to\infty}\int_{r=0}^{b} re^{-\frac{r^2}{2}}\,dr$$

Need to find $A = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx$

$$A^2 = \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{-x^2-y^2}{2}}\, dy\, dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}}\, dy\, dx$$

**Switch To Polar Coordinates:** $A^2 = \int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} e^{-\frac{r^2}{2}}\, r\, d\theta\, dr$

$$A^2 = 2\pi \int_{r=0}^{\infty} re^{-\frac{r^2}{2}}\, dr = 2\pi \lim_{b\to\infty}\int_{r=0}^{b} re^{-\frac{r^2}{2}}\, dr$$

$$= 2\pi \lim_{b\to\infty}\left[-e^{-\frac{r^2}{2}}\right]_0^b = 2\pi \lim_{b\to\infty}\left[-\frac{1}{e^{b^2/2}} + \frac{1}{e^0}\right] = 2\pi \times 1 = 2\pi$$

Need to find $A = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx$

$$A^2 = \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{-x^2-y^2}{2}}\, dy\, dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}}\, dy\, dx$$

**Switch To Polar Coordinates:** $A^2 = \int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} e^{-\frac{r^2}{2}}\, r\, d\theta\, dr$

$$A^2 = 2\pi \int_{r=0}^{\infty} re^{-\frac{r^2}{2}}\, dr = 2\pi \lim_{b\to\infty} \int_{r=0}^{b} re^{-\frac{r^2}{2}}\, dr$$

$$= 2\pi \lim_{b\to\infty} \left[-e^{-\frac{r^2}{2}}\right]_0^b = 2\pi \lim_{b\to\infty}\left[-\frac{1}{e^{b^2/2}} + \frac{1}{e^0}\right] = 2\pi \times 1 = 2\pi$$

Thus $A^2 = 2\pi$ so $A = \sqrt{2\pi}$

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \, dx = \sqrt{2\pi}$$

To get a probability density, let $p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$
This density is called the **Standard Normal Density**

Example: Suppose two numbers $b$ and $c$ are chosen at random between 0 and 1.

Example: Suppose two numbers $b$ and $c$ are chosen at random between 0 and 1.

What is the probability that the quadratic equation $x^2 + bx + c = 0$ has a real root?

Example: Suppose two numbers $b$ and $c$ are chosen at random between 0 and 1.

What is the probability that the quadratic equation $x^2 + bx + c = 0$ has a real root?

Solution: Choosing $b$ and $c$ is equivalent to choosing a point $(b, c)$ from the unit square $S$ with $p(\vec{x}) = 1$ ( **Uniform Density**)

Example: Suppose two numbers $b$ and $c$ are chosen at random between 0 and 1.

What is the probability that the quadratic equation $x^2 + bx + c = 0$ has a real root?

Solution: Choosing $b$ and $c$ is equivalent to choosing a point $(b, c)$ from the unit square $S$ with $p(\vec{x}) = 1$ ( **Uniform Density**)

Then $\int_S p(\vec{x}) = \int_S 1 = area(S) = 1$.

Example: Suppose two numbers $b$ and $c$ are chosen at random between 0 and 1.

What is the probability that the quadratic equation $x^2 + bx + c = 0$ has a real root?

Solution: Choosing $b$ and $c$ is equivalent to choosing a point $(b, c)$ from the unit square $S$ with $p(\vec{x}) = 1$ ( **Uniform Density**)

Then $\int_S p(\vec{x}) = \int_S 1 = area(S) = 1$.

Now $x^2 + bx + c = 0$ has solution $x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$

Example: Suppose two numbers $b$ and $c$ are chosen at random between 0 and 1.

What is the probability that the quadratic equation $x^2 + bx + c = 0$ has a real root?

Solution: Choosing $b$ and $c$ is equivalent to choosing a point $(b, c)$ from the unit square $S$ with $p(\vec{x}) = 1$ ( **Uniform Density**)

Then $\int_S p(\vec{x}) = \int_S 1 = area(S) = 1$.

Now $x^2 + bx + c = 0$ has solution $x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$

For real root, need $b^2 - 4c \geq 0$ or $c \leq \frac{b^2}{4}$

Example: Suppose two numbers $b$ and $c$ are chosen at random between 0 and 1.

What is the probability that the quadratic equation $x^2 + bx + c = 0$ has a real root?

Solution: Choosing $b$ and $c$ is equivalent to choosing a point $(b, c)$ from the unit square $S$ with $p(\vec{x}) = 1$ ( **Uniform Density**)

Then $\int_S p(\vec{x}) = \int_S 1 = area(S) = 1.$

Now $x^2 + bx + c = 0$ has solution $x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$

For real root, need $b^2 - 4c \geq 0$ or $c \leq \frac{b^2}{4}$

Let $T = \{(b, c) : c \leq \frac{b^2}{4}\}$

Example: Suppose two numbers $b$ and $c$ are chosen at random between 0 and 1.

What is the probability that the quadratic equation $x^2 + bx + c = 0$ has a real root?

Solution: Choosing $b$ and $c$ is equivalent to choosing a point $(b, c)$ from the unit square $S$ with $p(\vec{x}) = 1$ ( **Uniform Density**)

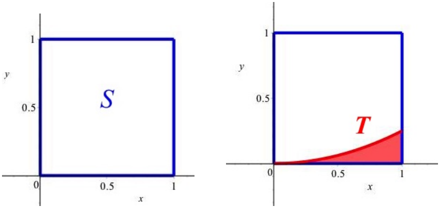
Then $\int_S p(\vec{x}) = \int_S 1 = area(S) = 1$.

Now $x^2 + bx + c = 0$ has solution $x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$

For real root, need $b^2 - 4c \geq 0$ or $c \leq \frac{b^2}{4}$

Let $T = \{(b, c) : c \leq \frac{b^2}{4}\}$

$$\int_T p(\vec{x}) = \int_{x=0}^{1} \int_{y=0}^{x^2/4} 1\, dy\, dx = \int_{x=0}^{1} \frac{x^2}{4}\, dx = \frac{x^3}{12}\Big|_0^1 = \frac{1}{12}$$



## General Exponential Probability Distribution

$$p(x) = \lambda e^{-\lambda x} \text{ for } x \geq 0, \lambda > 0$$

# General Exponential Probability Distribution

$$p(x) = \lambda e^{-\lambda x} \text{ for } x \geq 0, \lambda > 0$$

Easy to Show:

$$\int_0^\infty \lambda e^{-\lambda x}\, dx = 1 \text{ so it is a probability distribution}$$

## General Exponential Probability Distribution

$$p(x) = \lambda e^{-\lambda x} \text{ for } x \geq 0, \lambda > 0$$

Easy to Show:

$$\int_0^\infty \lambda e^{-\lambda x}\, dx = 1 \text{ so it is a probability distribution}$$

$$\text{Mean } \int_0^\infty \lambda x e^{-\lambda x}\, dx = \frac{1}{\lambda}$$

# General Exponential Probability Distribution

$$p(x) = \lambda e^{-\lambda x} \text{ for } x \geq 0, \lambda > 0$$

Easy to Show:

$$\int_0^\infty \lambda e^{-\lambda x}\, dx = 1 \text{ so it is a probability distribution}$$

$$\text{Mean } \int_0^\infty \lambda x e^{-\lambda x}\, dx = \frac{1}{\lambda}$$

$$\text{Prob(Bulb life} \geq 3) = 1 - \int_3^\infty \lambda e^{-\lambda x} dx = 1 + e^{-\lambda x}\Big|_3^\infty = 1 - e^{-3\lambda}$$

## General Exponential Probability Distribution

$$p(x) = \lambda e^{-\lambda x} \text{ for } x \geq 0, \lambda > 0$$

Easy to Show:

$$\int_0^\infty \lambda e^{-\lambda x}\, dx = 1 \text{ so it is a probability distribution}$$

$$\text{Mean } \int_0^\infty \lambda x e^{-\lambda x}\, dx = \frac{1}{\lambda}$$

$$\text{Prob(Bulb life} \geq 3) = 1 - \int_3^\infty \lambda e^{-\lambda x} dx = 1 + e^{-\lambda x}\Big|_3^\infty = 1 - e^{-3\lambda}$$

$$\text{Prob(2 lights have life} \geq 3) = e^{-3\lambda}(1 + 3\lambda)$$

# General Exponential Probability Distribution

$$p(x) = \lambda e^{-\lambda x} \text{ for } x \geq 0, \lambda > 0$$

Easy to Show:

$$\int_0^\infty \lambda e^{-\lambda x}\, dx = 1 \text{ so it is a probability distribution}$$

$$\text{Mean } \int_0^\infty \lambda x e^{-\lambda x}\, dx = \frac{1}{\lambda}$$

$$\text{Prob(Bulb life} \geq 3) = 1 - \int_3^\infty \lambda e^{-\lambda x} dx = 1 + e^{-\lambda x}\Big|_3^\infty = 1 - e^{-3\lambda}$$

$$\text{Prob(2 lights have life} \geq 3) = e^{-3\lambda}(1 + 3\lambda)$$

More than $b$ hours: $e^{-3b\lambda}(1 + b\lambda)$