MATH 224: Vector Calculus

Notes on Class 3

February 13, 2026


Valentine's Day
2026

Handouts

Notes on Assignment 2
Assignment 3

Derivatives and Integrals for $\mathbf{f} : \mathcal{R}^1 \to \mathcal{R}^n$

Vector-Valued Functions of a Real Variable

Begin with $\mathbf{F} : \mathcal{R}^1 \to \mathcal{R}^2$

$$\mathbf{F}(x) = (f(x),\ g(x))$$

Difference Quotient

$$\frac{\mathbf{F}(x+h) - \mathbf{F}(x)}{h} = \left(\frac{f(x+h) - f(x)}{h}, \frac{g(x+h) - g(x)}{h} \right)$$

So

$$\mathbf{F}'(x) = \lim_{h \to 0} \frac{\mathbf{F}(x+h) - \mathbf{F}(x)}{h} = (f'(x), g'(x))$$

Example: $\mathbf{F}(x) = (\cos x, x^3 - 2x)$
Solution: $\mathbf{F}'(x) = (-\sin x, 3x^2 - 2)$

Example: $\mathbf{F}(t) = (\tan t, \ln t)$
Solution: $\mathbf{F}'(t) = (\sec^2 t, \frac{1}{t})$

Nothing Special about $m = 2$

$$\mathbf{F}(x) = (f_1(x), f_2(x), ..., f_m(x))$$

$$\mathbf{F}'(x) = \left(f_1'(x), f_2'(x), ..., f_m'(x)\right)$$

Example: $\mathbf{F}(t) = (t^7, t^{-3}, \sin(t^2))$

Derivative: $(7t^6, -3t^{-4}, 2t\cos(t^2))$

IMAGE of **F** is a Curve (1 dimensional) in $\mathcal{R}^m$.

Tangent Lines

$$\mathbf{L}(t) = \mathbf{F}(x) + t\mathbf{F}'(x)$$

Example: $\mathbf{F}(x) = (x^3 + 7x + 3, 8 + \sin x)$
Then $\mathbf{F}'(x) = (3x^2 + 7, \cos x)$
At $x = 0 : \mathbf{F}(0) = (3, 8)$ and $\mathbf{F}'(0) = (7, 1)$
The Equation for the tangent line at (3,8) is

$$\mathbf{L}(t) = (3, 8) + t(7, 1) = (3 + 7t, 8 + t)$$

We can write as $x = 3 + 7t, y = 8 + t$ so
$t = \frac{x-3}{7}$ and $y = 8 + \frac{x-3}{7}$

Bottom Line

$$\mathbf{F} = (f_1, f_2, ..., f_m)$$

where each $f_i : \mathcal{R}^1 \rightarrow \mathcal{R}^1$

$\mathbf{F}$ is continuous if and only if each f_i is continuous

$\mathbf{F}$ is differentiable if and only if each f_i is differentiable

$$\mathbf{F}' = (f_1', f_2', ..., f_m')$$

$$\int \mathbf{F} = \left(\int f_1, \int f_2, ..., \int f_m \right)$$

KEY STEP IS THEOREM 2.2.1

THEOREM 2.2.1

If $\mathbf{f} : \mathcal{R}^1 \rightarrow \mathcal{R}^m$ with coordinate functions $f_1, f_2, ..., f_m$ and $\mathbf{L} = \{L_1, L_2, ..., L_m\}$ is a vector in $\mathcal{R}^m$, then

$$\lim_{x \rightarrow a} \mathbf{f}(x) = \mathbf{L} \text{ if and only if } \lim_{x \rightarrow a} f_j(x) = L_j \text{ for } j = 1, ..m$$

Suppose $\mathbf{p}$ and $\mathbf{q}$ are differentiable vector valued functions of a real number. Show

$$(\mathbf{p}\cdot\mathbf{q})' = \mathbf{p}'\cdot\mathbf{q} + \mathbf{p}\cdot\mathbf{q}'$$

Proof: For simplicity of notation, let's assume $\mathbf{p} = (P_1, P_2, P_3)$ and $\mathbf{q} = (Q_1, Q_2, Q_3)$ where the $P's$ and $Q's$ are dfferentiable real-valued functions of a real number. Then

$$\begin{aligned}
(\mathbf{p}\cdot\mathbf{q})' = \quad & ((P_1, P_2, P_3)\cdot(Q_1, Q_2, Q_3))' \\
= \quad & (P_1Q_1 + P_2Q_2 + P_3Q_3)' \\
= \quad & (P_1Q_1)' + (P_2Q_2)' + (P_3Q_3)' \\
= \quad & P_1'Q_1 + P_1Q_1' + P_2'Q_2 + P_2Q_2' + P_3'Q_3 + P_3Q_3' \\
= \quad & P_1'Q_1 + P_2'Q_2 + P_3'Q_3 + P_1Q_1' + +P_2Q_2' + +P_3Q_3' \\
= \quad & (P_1, P_2, P_3)'\cdot(Q_1, Q_2, Q_3) + (P_1, P_2, P_3)\cdot(Q_1', Q_2', Q_3') \\
= \quad & \mathbf{p}'\cdot\mathbf{q} + \mathbf{p}\cdot\mathbf{q}'
\end{aligned}$$