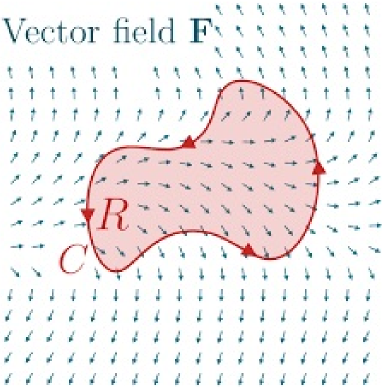
MATH 224: Vector Calculus



Class 32: Friday, May 1 2026

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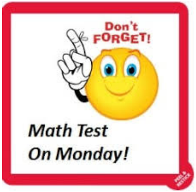
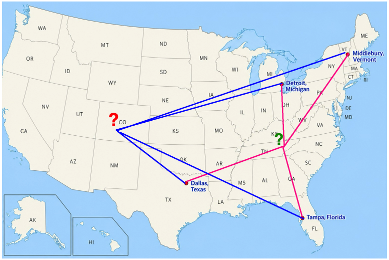
Notes on Assignment 29
Assignment 30
Green'sTheorem



7 PM – ?
Room:

Preparing for a Monday exam requires capitalizing on the weekend for focused review while ensuring adequate rest to maintain high performance. Focus on active recall, such as practice questions and reviewing chapter summaries, rather than passive re-reading. Prioritize high-weight topics, get sufficient sleep for memory consolidation, and avoid all-nighters.



Team Project
Minimizing Travel Costs
Due: Friday, May 8

Final Exam
Friday, May 15
9 AM – Noon

Announcements

Today

More Green's Theorem

Conservative Vector Fields

Divergence of a Vector Field

Definition div **F** $=$ trace of **F'**, the Jacobi Matrix

In general, div **F** is a real -valued function of n variables.

Curl of a Vector Field

Curl measures local tendency of a vector field and its flow lines to circulate around some axis.

The curl of a vector field is itself a vector field.

Setting; $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is our vector field

$\mathbf{F} = (F_1, F_2, F_3)$ so $\mathbf{F}(x, yz) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$

Formal Definition: curl $\mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$

Mnemonic Device:

$$\text{curl } \mathbf{F} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}. \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix}$$

Expand along first row:

$$\text{curl } \mathbf{F} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_2 & F_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ F_1 & F_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 \end{vmatrix} \mathbf{k}$$

Scalar Curl for Vector Fields in Plane

$\mathbf{F} = (F, G, 0)$ where $F(x, y)$ and $G(x, y)$ are functions only of x and y.

Then curl $\mathbf{F} = (0, 0, G_x - F_y)$

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Suppose **F** $= (F, G, 0)$ is gradient field with **F** $= \nabla f$.

Then $F = f_x$ and $G = f_y$

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Suppose $\mathbf{F} = (F, G, 0)$ is gradient field with $\mathbf{F} = \nabla f$.

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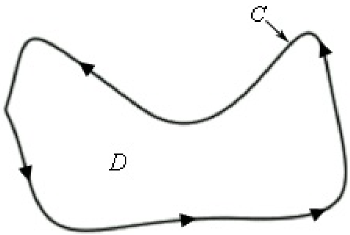
by Clairault's Theorem on Equality of Mixed Partials.

Green's Theorem in the Plane

$$\iint_D \text{curl}\,\mathbf{F} = \int_\gamma \mathbf{F}$$



D is bounded plane region.

$C = \gamma$ is piecewise smooth boundary of D

F and G are continuously differentiable functions defined on D

Then

$$\int\int (G_x - F_y)dxdy = \int_\gamma (F, G)$$

where γ is parametrized so it is traced once with D on the left.

Application of Green's Theorem in the Plane

$\iint_D \text{curl } \mathbf{F} = \int_\gamma \mathbf{F}$

Example $\mathbf{F}(x, y) = (0, x)$ implies $curl| : \mathbf{F} = 1 - 0 = 1$

Hence $\iint_D \text{curl } \mathbf{F} = \iint_D 1 =$ area of D

Green's Theorem enables us to find the area of a planar region if we can develop a parametrization of its boundary.

Example Consider the unit disk D of radius r centered at origin.

Let $g(t) = (r \cos t, r \sin t), 0 \leq t \leq 2\pi$

So $g'(t) = (r \sin t, r \cos t)$

and $\mathbf{F}(g(t)) = (0, r \cos t)$

Then $\mathbf{F}(g(t)) \cdot g'(t) = r^2 \cos^2 t \, dt$

Thus area of $D = \iint_D 1 = \iint_D \text{curl}\mathbf{F} = \int_\gamma \mathbf{F} = \int_0^{2\pi} r^2 \cos^2 t \, dt$

$\int_0^{2\pi} r^2 \cos^2 t \, dt = r^2 \int_0^{2\pi} \frac{1+\cos 2t}{2} \, dt = \frac{r^2}{2} \left[t + \frac{1}{2} \sin 2t\right]_0^{2=\pi} = \pi r^2$

Using Green's Theorem

(1) Compute $\iint_D \operatorname{curl} \mathbf{F}$ by using $\int_\gamma \mathbf{F}$

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Using Green's Theorem

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<u>Example</u> Let $\mathbf{F}(x, y) = (\frac{1}{y}\cos\frac{x}{y}, -\frac{x}{y^2}\cos\frac{x}{y})$

Using Green's Theorem

Compute $\int_\gamma \mathbf{F}$ by using $\iint_D \text{curl } \mathbf{F}$

Example Let $\mathbf{F}(x, y) = (\frac{1}{y} \cos \frac{x}{y}, -\frac{x}{y^2} \cos \frac{x}{y})$

Compute $\int_\gamma \mathbf{F}$ as $\iint_D (G_x - F_y)$

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So $G_x - F_y = 0$.

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So $G_x - F_y = 0$.

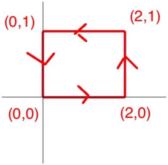
Hence $\int_\gamma \mathbf{F} = 0$

Example

Find

$$\int_\gamma (1+10xy+y^2)\,dx+(6xy+5x^2)\,dy = \int_\gamma (1+10xy+y^2, 6xy+5x^2)$$

where γ is boundary of the rectangle with vertices (0,0), (2,0), (2,1), and (0,1).



Note: Direct Computation requires 4 integrals.

$$F(x,y) = 1 + 10xy + y^2. \quad G(x,y) = 6xy + 5x^2$$

$$F_y = 10x + 2y \qquad . \; G_x = 6y + 10x$$

$$G_x - F_y = 6y + 10x - 10x - 2y = 4t$$

$$\int_\gamma \mathbf{F} = \iint_D \text{curl}\,\mathbf{F} = \int_0^2 \int_0^1 4y\,dy\,dx = \int_0^2 \left[2y^2\right]_0^1 = \int_0^2 2dx = 4$$

George Green
1793 – 1841

Mikhail Ostrogradsky
1801 – 1861

AN ESSAY

ON THE

APPLICATION

MATHEMATICAL ANALYSIS TO THE THEORIES OF
ELECTRICITY AND MAGNETISM.

Gauss' Theorem

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Observe $(g_2^{'}, -g_1^{'}) \cdot (g_1^{'}, g_2^{'}) = g_1^{'} g_2^{'} - g_1^{'} g_2^{'} = 0$

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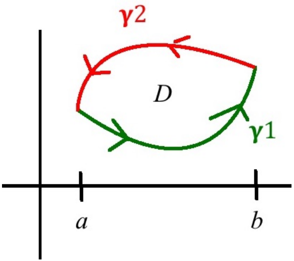
Putting everything together: $\boxed{\iint_D \text{div } \mathbf{F} = \int_\gamma \mathbf{H} = \int_\gamma \mathbf{F} \cdot \mathbf{N}}$

Proof of Green's Theorem in an Elementary Case

Case : Boundary of D is made up of the graphs of two functions defined on interval $[a, b]$.



Ingredients:

Vector Field $\mathbf{F} = (F, G) = (F, 0) + (0, G)$

$\gamma_1 =$ image of g_1

$\gamma_2 =$ image of g_2

Need to show $\iint_D [G_x - Fy] = \int_\gamma \mathbf{F} = \int_\gamma [(F, 0) + (0, G)]$

Will show $\iint_D -Fy = \int_\gamma (F, 0)$

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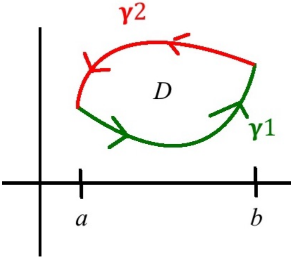
We tackle the line integral first. Start with γ_1



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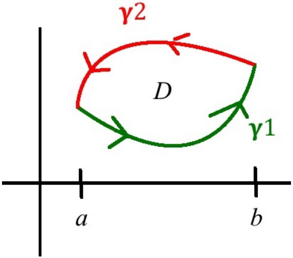

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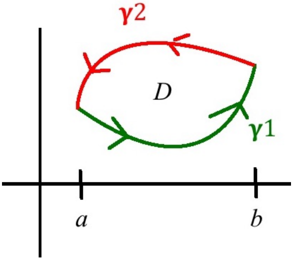

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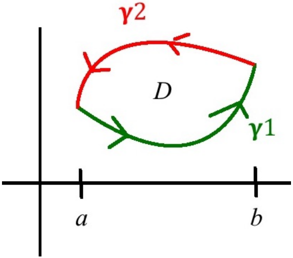

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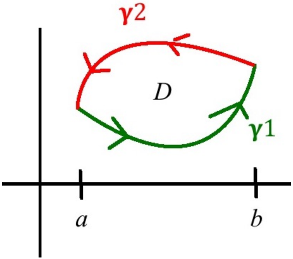

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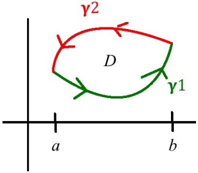

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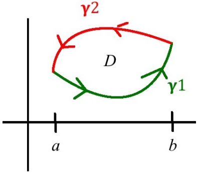

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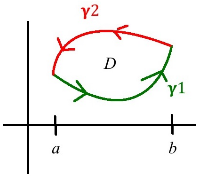

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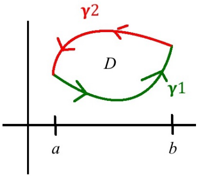

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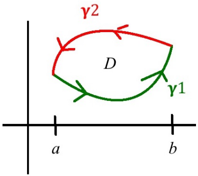

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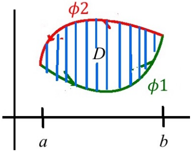

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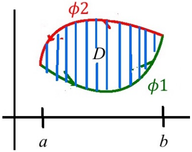

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Conservative Vector Fields

F is continuously differentiable vector field in the plane

F : $\mathbb{R}^2 \to \mathbb{R}^2$ with $\mathbf{F}(x, y) = (F(x, y), G(x, y))$ where F and G are each real-valued functions.

Here curl **F** is a real-valued function $G_x - F_y$

Green's Theorem: $\int_D \text{curl } \mathbf{F} = \int_\gamma \mathbf{F}$

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Three Important Properties of Vector Fields

A: **F** is **CONSERVATIVE** means $\mathbf{F} = \nabla f$ for some $f : \mathbb{R}^2 \to \mathbb{R}^1$

B: **F** is **IRROTATIONAL** means curl $\mathbf{F} = 0$

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Major Goal: Show THESE PROPERTIES ARE EQUIVALENT

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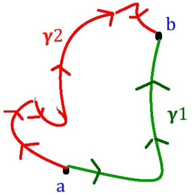

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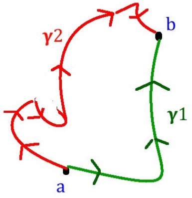

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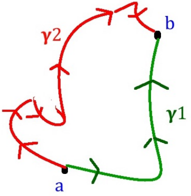

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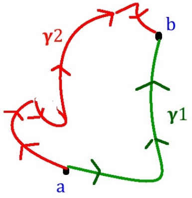

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