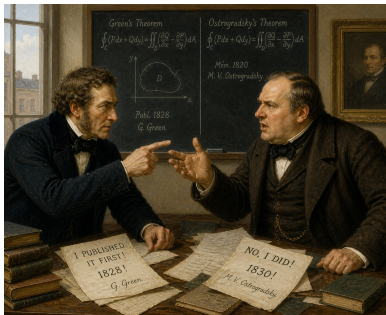


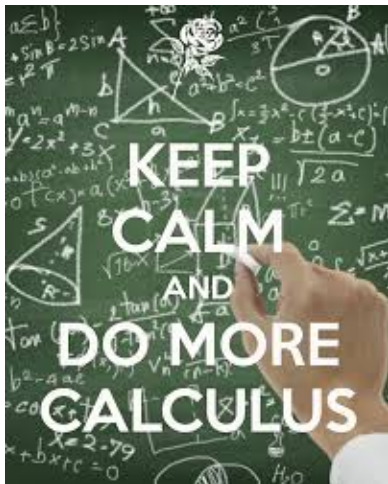
MATH 224: Vector Calculus



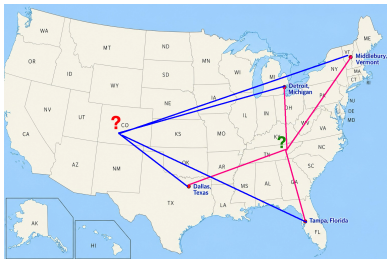
Class 33: Monday, May 4, 2026



Green's Theorem



Exam 3
Tonight Starting at 7 PM
Warner 105



Team Project
Minimizing Travel Costs
Due: Friday, May 8

Course Response Forms
In Class Next Monday

Final Exam
Friday, May 15
9 AM – Noon

Announcements

Today

More Green's Theorem
Conservative Vector Fields

Outline

- ▶ Definition and Calculation of Curl
- ▶ Scalar Curl in the Plane
- ▶ Green's Theorem in the Plane
- ▶ Gauss's Theorem
- ▶ Proving Green's Theorem
- ▶ Conservative Vector Fields

Curl of a Vector Field

The curl of a vector field is itself a vector field.

Setting; $\mathbf{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is our vector field

$$\mathbf{F} = (F_1, F_2, F_3) \text{ so } \mathbf{F}(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$$

$$\text{Formal Definition: } \text{curl } \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

$$\text{Mnemonic Device: } \text{curl } \mathbf{F} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix}$$

Scalar Curl for Vector Fields in Plane

$\mathbf{F} = (F, G, 0)$ where $F(x, y)$ and $G(x, y)$ are functions only of x and y .

$$\text{Then curl } \mathbf{F} = (0, 0, G_x - F_y)$$

Note: Curl and Conservative Vector Field

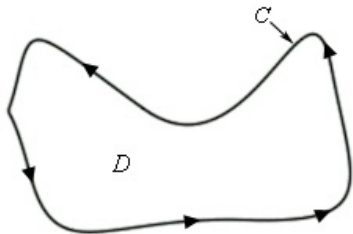
Suppose $\mathbf{F} = (F, G, 0)$ is gradient field with $\mathbf{F} = \nabla f$.

$$\text{Then } F = f_x \text{ and } G = f_y$$

In this case, $\text{Curl } \mathbf{F} = (0, 0, f_{yx} - f_{xy}) = (0, 0, 0)$
by Clairaut's Theorem on Equality of Mixed Partial.

Green's Theorem in the Plane

$$\iint_D \text{curl } \mathbf{F} = \int_{\gamma} \mathbf{F}$$



D is bounded plane region.

$C = \gamma$ is piecewise smooth boundary of D

F and G are continuously differentiable functions defined on D

Then

$$\int \int (G_x - F_y) dx dy = \int_{\gamma} (F, G)$$

where γ is parametrized so it is traced once with D on the left.

Application of Green's Theorem in the Plane

$$\iint_D \text{curl } \mathbf{F} = \int_{\gamma} \mathbf{F}$$

Example $\mathbf{F}(x, y) = (0, x)$ implies $\text{curl} \mathbf{F} = 1 - 0 = 1$

Hence $\iint_D \text{curl } \mathbf{F} = \iint_D 1 = \text{area of } D$

Green's Theorem enables us to find the area of a planar region if we can develop a parametrization of its boundary.

Example Consider the unit disk D of radius r centered at origin.

Let $g(t) = (r \cos t, r \sin t), 0 \leq t \leq 2\pi$

So $g'(t) = (-r \sin t, r \cos t)$

and $\mathbf{F}(g(t)) = (0, r \cos t)$

Then $\mathbf{F}(g(t)) \cdot g'(t) = r^2 \cos^2 t$

Thus area of $D = \iint_D 1 = \iint_D \text{curl } \mathbf{F} = \int_{\gamma} \mathbf{F} = \int_0^{2\pi} r^2 \cos^2 t \, dt$
 $\int_0^{2\pi} r^2 \cos^2 t \, dt = r^2 \int_0^{2\pi} \frac{1+\cos 2t}{2} \, dt = \frac{r^2}{2} \left[t + \frac{1}{2} \sin 2t \right]_0^{2\pi} = \pi r^2$

Using Green's Theorem

(1) Compute $\iint_D \text{curl } \mathbf{F}$ by using $\int_\gamma \mathbf{F}$

(2) Compute $\int_\gamma \mathbf{F}$ by using $\iint_D \text{curl } \mathbf{F}$

Using Green's Theorem

Compute $\int_{\gamma} \mathbf{F}$ by using $\iint_D \text{curl } \mathbf{F}$

Example Let $\mathbf{F}(x, y) = \left(\frac{1}{y} \cos \frac{x}{y}, -\frac{x}{y^2} \cos \frac{x}{y}\right)$

Compute $\int_{\gamma} \mathbf{F}$ as $\iint_D (G_x - F_y)$

Here $G_x = \left(-\frac{x}{y^2}\right)_x \cos \frac{x}{y} + -\frac{x}{y^2} \left(\cos \frac{x}{y}\right)_x$

$$\begin{aligned} &= -\frac{1}{y^2} \cos \frac{x}{y} - \frac{x}{y^2} \left(-\sin \frac{x}{y}\right) \left(\frac{1}{y}\right) \\ &= -\frac{1}{y^2} \cos \frac{x}{y} + \frac{x}{y^3} \left(\sin \frac{x}{y}\right) \end{aligned}$$

Similarly, $F_y = -\frac{1}{y^2} \cos \frac{x}{y} + \frac{1}{y} \left(-\sin \frac{x}{y}\right) \left(\frac{-x}{y^2}\right)$

$$= -\frac{1}{y^2} \cos \frac{x}{y} + \frac{x}{y^3} \left(+\sin \frac{x}{y}\right)$$

So $G_x - F_y = 0$.

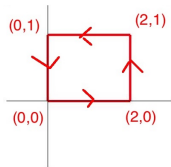
Hence $\int_{\gamma} \mathbf{F} = 0$

Example

Find

$$\int_{\gamma} (1+10xy+y^2)dx + (6xy+5x^2)dy = \int_{\gamma} (1+10xy+y^2, 6xy+5x^2)$$

where γ is boundary of the rectangle with vertices $(0,0)$, $(2,0)$, $(2,1)$, and $(0,1)$.



Note: Direct Computation requires 4 integrals.

$$F(x, y) = 1 + 10xy + y^2. \quad G(x, y) = 6xy + 5x^2$$

$$F_y = 10x + 2y \quad . \quad G_x = 6y + 10x$$

$$G_x - F_y = 6y + 10x - 10x - 2y = 4y$$

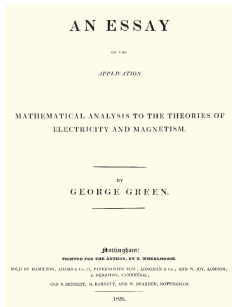
$$\int_{\gamma} \mathbf{F} = \iint_D \text{curl } \mathbf{F} = \int_0^2 \int_0^1 4y \, dy \, dx = \int_0^2 [2y^2]_0^1 = \int_0^2 2 \, dx = 4$$

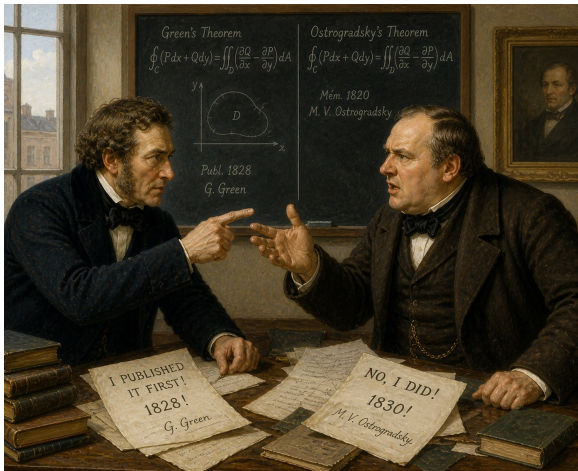


George Green
1793 – 1841



Mikhail Ostrogradsky
1801 – 1861





- ▶ Joseph-Louis Lagrange (1762): First formulated the foundational concept.
- ▶ Carl Friedrich Gauss (1813): Independently developed the divergence theorem while working on electromagnetism but left it unpublished.
- ▶ Mikhail Ostrogradsky (1828): The first to rigorously prove and publish the theorem, cementing his rightful place in its namesake.
- ▶ George Green (1828): Independently discovered the theorem while working on electricity and magnetism, although his work initially went largely unnoticed.
- ▶ Siméon-Denis Poisson (1824) and Pierre-Frédéric Sarrus (1828): Also contributed to discovering and proving various forms of the theorem.

Gauss' Theorem

$$\text{Green: } \iint_D \text{curl } \mathbf{F} = \int_{\gamma} \mathbf{F}$$

$$\text{If } \mathbf{F} = (F_1, F_2) \text{ then } \text{curl } \mathbf{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

Apply Green's Theorem to $\mathbf{H} = (-G, F)$ where $\mathbf{F} = (F, G)$

$$\int_{\gamma} \mathbf{H} = \iint_D \text{curl } (F_x - (-G_y)) = \iint_D (F_x + G_y) = \iint_D \text{div } \mathbf{F} \text{ On}$$

the other hand, $\int_{\gamma} \mathbf{H} = \int_a^b \mathbf{H} \cdot \mathbf{g}' = \int_a^b (-G, F) \cdot (g'_1, g'_2)$

$$\int_a^b (-G, F) \cdot (g'_1, g'_2) = \int_a^b -Gg'_1 + Fg'_2 = \int_a^b (F, G) \cdot (g'_2, -g'_1)$$

$$\text{Observe } (g'_2, -g'_1) \cdot (g'_1, g'_2) = g'_1g'_2 - g'_1g'_2 = 0$$

So $(g'_2, -g'_1)$ is orthogonal to the tangent vector so it is a normal vector \mathbf{N} .

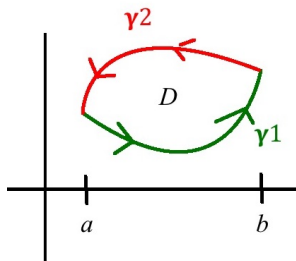
$$\text{Thus } \int_{\gamma} \mathbf{H} = \int_a^b (F, G) \cdot (g'_2, -g'_1) = \int_a^b (F, G) \cdot \mathbf{N} = \int_{\gamma} \mathbf{F} \cdot \mathbf{N}$$

Putting everything together:

$$\boxed{\iint_D \text{div } \mathbf{F} = \int_{\gamma} \mathbf{H} = \int_{\gamma} \mathbf{F} \cdot \mathbf{N}}$$

Proof of Green's Theorem in an Elementary Case

Case : Boundary of D is made up of the graphs of two functions defined on interval $[a, b]$.



Ingredients:

Vector Field $\mathbf{F} = (F, G) = (F, 0) + (0, G)$

$\gamma_1 = \text{image of } g_1$

$\gamma_2 = \text{image of } g_2$

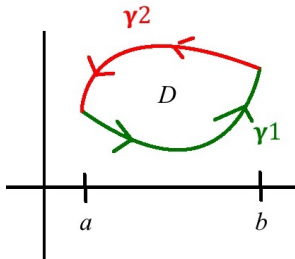
Need to show $\iint_D [G_x - F_y] = \int_{\gamma} \mathbf{F} = \int_{\gamma} [(F, 0) + (0, G)]$

Will show $\iint_D -F_y = \int_{\gamma} (F, 0)$

Need to show $\iint_D [G_x - Fy] = \int_\gamma \mathbf{F} = \int_\gamma [(F, 0) + (0, G)]$

Will show $\iint_D -Fy = \int_\gamma (F, 0)$

We tackle the line integral first. Start with γ_1



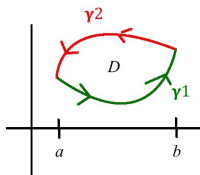
We can parametrize γ_1 by a function $g(t) = (t, \phi(t))$ for $a \leq t \leq b$

Then $g'(t) = (1, \phi_1'(t))$

Now $(F, 0) \cdot g'(t) = (F, 0) \cdot (1, \phi_1'(t)) = F = F(t, \phi_1(t))$

so $\int_{\gamma_1} (F, 0) = \int_a^b F(t, \phi_1(t)) dt$

Now we take up γ_2



Consider Parametrization of γ_2 as $g(t) = (t, \phi_2(t))$, $a \leq t \leq b$.
This would actually traces out γ_2 in the opposite direction. It is
the parametrization of $-\gamma_2$

Again we have $g'(t) = (1, \phi_2')$ and $(F, 0) \cdot g'(t) = F(t, \phi_2(t))$
so $\int_{-\gamma_2} (F, 0) = \int_a^b F(t, \phi_2(t))$.

Thus $\int_{-\gamma_2} (F, 0) = - \int_{\gamma_2} = - \int_a^b F(t, \phi_2(t))$.

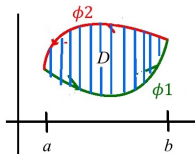
Finally, $\int_{\gamma} (F, 0) = \int_{\gamma_1} (F, 0) + \int_{\gamma_2} (F, 0)$
 $= \int_a^b F(t, \phi_1(t)) dt - \int_a^b F(t, \phi_2(t)) dt$

$$\int_{\gamma} (F, 0) = \int_a^b F(t, \phi_1(t)) - F(t, \phi_2(t)) dt$$

Goal: Show $\iint_D -F_y = \int_\gamma (F, 0)$

So far: $\int_\gamma (F, 0) = \int_a^b F(t, \phi_1(t)) - F(t, \phi_2(t)) dt$

Now turn to the curl part:



$$\begin{aligned}\iint_D -F_y &= - \iint_D F_y = \int_{x=a}^{x=b} \int_{y=\phi_1(x)}^{y=\phi_2(x)} -F_y(x, y) dy dx \\ &= - \int_a^b [F(x, \phi_2(x)) - F(x, \phi_1(x))] dx \\ &= - \int_a^b [F(t, \phi_2(t)) - F(t, \phi_1(t))] dt \text{ (let } t = x) \\ &= \int_a^b [F(t, \phi_1(t)) - F(t, \phi_2(t))] dt\end{aligned}$$

Conservative Vector Fields

\mathbf{F} is continuously differentiable vector field in the plane

$\mathbf{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with $\mathbf{F}(x, y) = (F(x, y), G(x, y))$ where F and G are each real-valued functions.

Here $\text{curl } \mathbf{F}$ is a real-valued function $G_x - F_y$

$$\text{Green's Theorem: } \int_D \text{curl } \mathbf{F} = \int_{\gamma} \mathbf{F}$$

Three Important Properties of Vector Fields

A: \mathbf{F} is **CONSERVATIVE** means $\mathbf{F} = \nabla f$ for some $f : \mathbb{R}^2 \rightarrow \mathbb{R}^1$

B: \mathbf{F} is **IRROTATIONAL** means $\text{curl } \mathbf{F} = 0$

C: \mathbf{F} is **PATH INDEPENDENT** means $\int_{\gamma_1} \mathbf{F} = \int_{\gamma_2} \mathbf{F}$ for any paths γ_1 and γ_2 from \mathbf{a} to \mathbf{b} where \mathbf{a} and \mathbf{b} are any points in the plane.

Major Goal: Show THESE PROPERTIES ARE EQUIVALENT

A implies B

A **F** is **CONSERVATIVE** means $\mathbf{F} = \nabla f$ for some $f : \mathbb{R}^2 \rightarrow \mathbb{R}^1$

B **F** is **IRROTATIONAL** means $\text{curl } \mathbf{F} = 0$

Suppose **F** is Conservative

Then $(F, G) = \mathbf{F} = \nabla f = (f_x, f_y)$ so $f_x = F$ and $f_y = G$

Then $G_x = f_{yx}$ and $F_y = f_{xy}$

so $\text{curl } \mathbf{F} = G_x - F_y = f_{yx} - f_{xy} = 0$

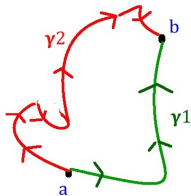
by equality of mixed partials.

B implies **C** will follow from Green's Theorem

B **F** is **IRROTATIONAL** means $\text{curl } \mathbf{F} = 0$

C **F** is **PATH INDEPENDENT** means $\int_{\gamma_1} \mathbf{F} = \int_{\gamma_2} \mathbf{F}$ for any paths γ_1 and γ_2 from **a** to **b** where **a** and **b** are any points in the plane.

Let **a** and **b** are any points in the plane and γ_1 and γ_2 two paths from **a** to **b**. Then $-\gamma_1$ runs from **b** to **a**



and $\gamma = \gamma_1 - \gamma_2$ is a loop that begins and ends at **a**

Let D be the enclosed region.

By Green's Theorem $\int_{\gamma} \mathbf{F} = \iint_D \text{curl } \mathbf{F} = \iint_D 0 = 0$

$$\text{Thus } 0 = \int_{\gamma} \mathbf{F} = \int_{\gamma_1 - \gamma_2} \mathbf{F} = \int_{\gamma_1} \mathbf{F} - \int_{\gamma_2} \mathbf{F}$$

$$\text{Hence } \int_{\gamma_2} \mathbf{F} = \int_{\gamma_1} \mathbf{F}$$

C implies **A**

- C** **F** is **PATH INDEPENDENT** means $\int_{\gamma_1} \mathbf{F} = \int_{\gamma_2} \mathbf{F}$ for any paths γ_1 and γ_2 from **a** to **b** where **a** and **b** are any points in the plane.
- A** **F** is **CONSERVATIVE** means $\mathbf{F} = \nabla f$ for some $f : \mathbb{R}^2 \rightarrow \mathbb{R}^1$