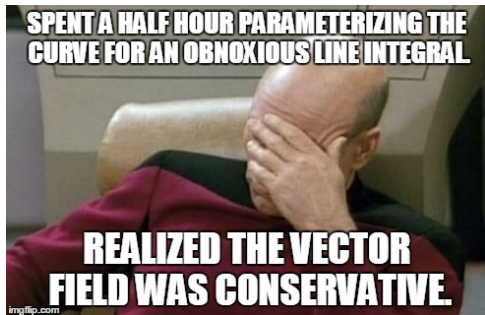


MATH 224: Vector Calculus



Class 34: Wednesday, May 6, 2026



Notes on Assignment 30
Assignment 31
Surface Integrals

Rough Weights for Course Components

Exam1	20%
Exam 2	20%
Exam 3	20%
Final Exam	30%
Project	10 %

Final Exam:
MATH 224: Friday, May 15 9 AM –Noon

Announcements

Location Problem Due Friday, May 8

Today

More About Conservative Vector Fields

Surface Integrals

Conservative Vector Fields

\mathbf{F} is continuously differentiable vector field in the plane

$\mathbf{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with $\mathbf{F}(x, y) = (F(x, y), G(x, y))$ where F and G are each real-valued functions.

Here $\text{curl } \mathbf{F}$ is a real-valued function $G_x - F_y$

$$\text{Green's Theorem: } \int_D \text{curl } \mathbf{F} = \int_{\gamma} \mathbf{F}$$

Three Important Properties of Vector Fields

A \mathbf{F} is **CONSERVATIVE** means $\mathbf{F} = \nabla f$ for some $f : \mathbb{R}^2 \rightarrow \mathbb{R}^1$

B \mathbf{F} is **IRROTATIONAL** means $\text{curl } \mathbf{F} = 0$

C \mathbf{F} is **PATH INDEPENDENT** means $\int_{\gamma_1} \mathbf{F} = \int_{\gamma_2} \mathbf{F}$ for any paths γ_1 and γ_2 from \mathbf{a} to \mathbf{b} where \mathbf{a} and \mathbf{b} are any points in the plane.

Major Goal: Show THESE PROPERTIES ARE EQUIVALENT

A implies **B**

A **F** is **CONSERVATIVE** means $\mathbf{F} = \nabla f$ for some $f : \mathbb{R}^2 \rightarrow \mathbb{R}^1$

B **F** is **IRROTATIONAL** means $\text{curl } \mathbf{F} = 0$

Suppose **F** is Conservative

Then $(F, G) = \mathbf{F} = \nabla f = (f_x, f_y)$ so $f_x = F$ and $f_y = G$

Thus $G_x = f_{yx}$ and $F_y = f_{xy}$

so $\text{curl } \mathbf{F} = G_x - F_y = f_{yx} - f_{xy} = 0$

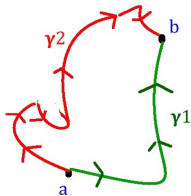
by equality of mixed partials.

B implies **C** will follow from Green's Theorem

B **F** is **IRROTATIONAL** means $\text{curl } \mathbf{F} = 0$

C **F** is **PATH INDEPENDENT** means $\int_{\gamma_1} \mathbf{F} = \int_{\gamma_2} \mathbf{F}$ for any paths γ_1 and γ_2 from **a** to **b** where **a** and **b** are any points in the plane.

Let **a** and **b** are any points in the plane and γ_1 and γ_2 two paths from **a** to **b**. Then $-\gamma_1$ runs from **b** to **a**



and $\gamma = \gamma_1 - \gamma_2$ is a loop that begins and ends at **a**

Let D be the enclosed region.

By Green's Theorem $\int_{\gamma} \mathbf{F} = \iint_D \text{curl } \mathbf{F} = \iint_D 0 = 0$

$$\text{Thus } 0 = \int_{\gamma} \mathbf{F} = \int_{\gamma_1 - \gamma_2} \mathbf{F} = \int_{\gamma_1} \mathbf{F} - \int_{\gamma_2} \mathbf{F}$$

$$\text{Hence } \int_{\gamma_2} \mathbf{F} = \int_{\gamma_1} \mathbf{F}$$

C implies **A**

C **F** is **PATH INDEPENDENT** means $\int_{\gamma_1} \mathbf{F} = \int_{\gamma_2} \mathbf{F}$ for any paths γ_1 and γ_2 from **a** to **b** where **a** and **b** are any points in the plane.

A **F** is **CONSERVATIVE** means $\mathbf{F} = \nabla f$ for some $f : \mathbb{R}^2 \rightarrow \mathbb{R}^1$

Idea:

Fix \mathbf{x}_0 in \mathbb{R}^n and let \mathbf{x} be arbitrary point in \mathbb{R}^n .

Let γ be a curve from \mathbf{x}_0 to \mathbf{x} .

Then $\int_{\gamma} \mathbf{F}$ will be a function of \mathbf{x} whose gradient is \mathbf{F} .

Theorem Let \mathbf{F} be a continuous vector field defined in a polygonally connected open set D of \mathbb{R}^n . If the line integral $\int_{\gamma} \mathbf{F}$ is independent of piecewise smooth path γ from \mathbf{x}_0 to \mathbf{x} in D , then if $f(\mathbf{x}) = \int_{\gamma} \mathbf{F}$, it is true that $\nabla f = \mathbf{F}$.

Example $\mathbf{F}(x, y) = (3x^2 + y, e^y + x)$

Here $\mathbf{F} = (F, G)$ so $F(x, y) = 3x^2 + y, G(x, y) = e^y + x$

Hence $F_y = 1, G_x = 1$ so $\text{curl } \mathbf{F} = G_x - F_y = 0$

Let's build f so its gradient $\nabla f = (f_x, f_y) = (3x^2 + y, e^y + x)$

We need $f_x = 3x^2 + y$ so do "partial integration with respect to x ":

$$f(x) = x^3 + yx + g(y). \quad [\text{Why is there } g(y)?]$$

Then $f_y = 0 + x + g'(y)$ which should equal $x + e^y$ so need

$$g'(y) = e^y$$

which we can get by letting $g(y) = e^y$.

Hence we can choose $f(x, y) = x^3 + yx + e^y + C$.

Let's build the potential function in a different way using the theorem with $\mathbf{F}(x, y) = (3x^2 + y, e^y + x)$

Pick $\mathbf{x}_0 = (0, 0)$ and let $\mathbf{x} = (x, y)$ be an arbitrary point. Choose the straight line between them as the path γ with parametrization

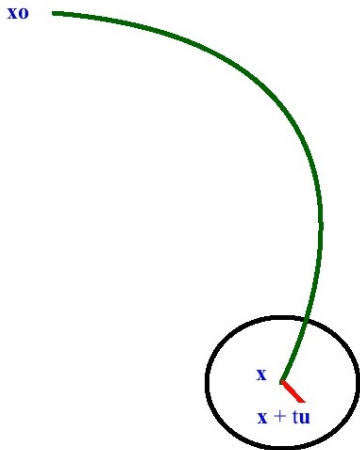
$$g(t) = (xt, yt), 0 \leq t \leq 1 \text{ so } g'(t) = (x, y)$$

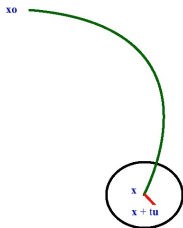
$$\text{Then } \mathbf{F}(g(t)) = F(xt, yt) = (3x^2t^2 + yt, e^{yt} + xt)$$

$$\begin{aligned} \text{so } \mathbf{F}(g(t)) \cdot g'(t) &= (3x^2t^2 + yt, e^{yt} + xt) \cdot (x, y) \\ &= 3x^3t^2 + xyt + ye^{yt} + xyt = 3x^3t^2 + 2xyt + ye^{yt} \end{aligned}$$

$$\begin{aligned} \text{Now } \int_{\gamma} \mathbf{F} &= \int_0^1 (3x^3t^2 + 2xyt + ye^{yt}) dt \\ &= [x^3t^3 + xyt^2 + e^{yt}]_{t=0}^{t=1} \\ &= (x^3 + xy + e^y) - (0 + 0 + 1) = x^3 + xy + e^y - 1 \end{aligned}$$

Theorem Let \mathbf{F} be a continuous vector field defined in a polygonally connected open set D of \mathbb{R}^n . If the line integral $\int_{\gamma} \mathbf{F}$ is independent of piecewise smooth path γ from \mathbf{x}_0 to \mathbf{x} in D , then if $f(\mathbf{x}) = \int_{\gamma} \mathbf{F}$, it is true that $\nabla f = \mathbf{F}$.





Let g be parametrization of line segment from \mathbf{x} to $\mathbf{x} + t\mathbf{u}$ so
 $g(v) = \mathbf{x} + v\mathbf{u}, 0 \leq v \leq t$ and $g'(v) = \mathbf{u}$

$$\begin{aligned} f(\mathbf{x} + t\mathbf{u}) - f(\mathbf{x}) &= \int_{\mathbf{x}_0}^{\mathbf{x} + t\mathbf{u}} \mathbf{F} - \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{F} = \int_{\mathbf{x}}^{\mathbf{x} + t\mathbf{u}} \mathbf{F}(\mathbf{x} + v\mathbf{u}) \\ &= \int_0^t \mathbf{F}(\mathbf{x} + v\mathbf{u}) \cdot \mathbf{u} \, dv \end{aligned}$$

To find $\frac{\partial f}{\partial x_j}(\mathbf{x})$, let \mathbf{u} be unit vector $\mathbf{e}_j = (0, 0, \dots, 1, 0, 0, \dots, 0)$
in the j th direction.

$$\begin{aligned}\frac{\partial f}{\partial x_j}(\mathbf{x}) &= \lim_{t \rightarrow 0} \frac{f(\mathbf{x} + t\mathbf{u}) - f(\mathbf{x})}{t} \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \int_0^t \mathbf{F}(\mathbf{x} + v\mathbf{u}) \cdot \mathbf{u} \, dv \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \int_0^t \mathbf{F}(\mathbf{x} + v\mathbf{e}_j) \cdot \mathbf{e}_j \, dv\end{aligned}$$

But this last expression is the derivative of the integral with respect to t evaluated at $t = 0$ which is $\mathbf{F} \cdot \mathbf{e}_j = F_j(\mathbf{x})$ (Using Fundamental Theorem of Calculus)

Symmetry of Jacobian Matrix for Conservative Vector Field

Let $\mathbf{F} = (F(x, y), G(x, y))$ be a conservative vector field in the plane which we can recognize by $G_x = F_y$

$$\mathbf{F}' = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} \text{ Note symmetry of Jacobian Matrix.}$$

How do things generalize to higher dimensions?

Example: $\mathbf{F}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by

$$F(x, y, z) = (yz^2 + \sin y + 3x^2, xz^2 + x \cos y + e^z, 2xyz + ye^z + \frac{1}{z})$$

$$\mathbf{F}' = \begin{pmatrix} 6x & z^2 + \cos y & 2yz \\ z^2 + \cos y & -x \sin y & 2xz + e^z \\ 2yz & 2xz + e^z & 2xy + ye^z - \frac{1}{z^2} \end{pmatrix}$$

To find f so that $\nabla f = \mathbf{F}$:

Step 1: integrate first component of \mathbf{F} with respect to x :

$$f(x, y, z) = yz^2x + x \sin y + x^3 + G(y, z)$$

Step 2: Take derivative of trial f respect to y and set equal to second component of \mathbf{F} :

$$f_y = z^2x + x \cos y + 0 + G_y(y, z) \text{ must } = xz^2 + x \cos y + e^z$$

$$\text{Need } G_y(y, z) = e^z \text{ so choose } G(y, z) = e^z y + H(z)$$

$$\text{So far, } f(x, y, z) = yz^2x + x \sin y + x^3 + e^z y + H(z)$$

Step 3: Take derivative of trial f respect to z and set equal to third component of \mathbf{F} ;

$$f_z(x, y, z) = 2xyz + 0 + 0 + e^z y + H'(z) \text{ must } = 2xyz + e^z y + \frac{1}{z}$$

$$\text{Need } H'(z) = \frac{1}{z} \text{ so choose } H(x) = \ln |z| + C \text{ Thus}$$

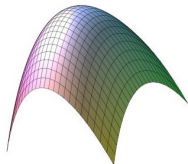
$$f(x, y, z) = f(x, y, z) = yz^2x + x \sin y + x^3 + e^z y + \ln |z| + C$$

Theorem If \mathbf{F} is a conservative vector field on \mathbb{R}^n and is continuously differentiable, then the Jacobian matrix is symmetric.

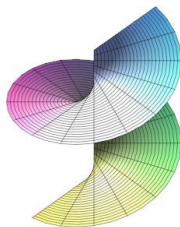
Proof: Equality of mixed partials.

Theorem Suppose \mathbf{F} is a continuously differentiable vector field on \mathbb{R}^n whose Jacobian matrix is symmetric. Then \mathbf{F} is conservative

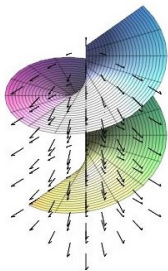
Integrating Vector Fields Over Surfaces



$$g(u, v) = [u, v, -2u^2 - 3v^2]$$



$$g(u, v) = [u \cos v, u \sin v, v]$$



Smooth Curve γ

$$g : I \text{ in } \mathbb{R}^1 \rightarrow \mathbb{R}^n$$

$$\text{Length} = \int_I |g'(t)| dt$$

$$\text{Mass} = \int_I \mu(g(t)) |g'(t)| dt$$

Line Integral:

$$\int_{\gamma} \mathbf{F} = \int_I \mathbf{F}(g(t)) \cdot g'(t) dt$$

Smooth Surface S

$$g : D \text{ in } \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

$$\text{Area } \sigma(S) = \iint_D |g_u \times g_v| dudv$$

$$\text{Mass} = \iint_D \mu d\sigma$$

Surface Integral

$$\iint_S \mathbf{F} = \iint_D \mathbf{F}(g(u, v)) \cdot (g_u \times g_v)$$

$$\iint_S \mathbf{F} = \iint_S \mathbf{F} \cdot dS = \iint_S \mathbf{F} \cdot \mathbf{N} d\sigma$$

$\Phi(\mathbf{F}, S) = \iint_S \mathbf{F}$ is **flux** of \mathbf{F} across S .

Surface Integral

Let g be a function from an interval $[t_0, t_1]$ into \mathbb{R}^n with image γ and m density at $g(t)$.

$$\text{Then Mass of Wire} = \int_{t_0}^{t_1} \mu(t) |g'(t)| dt$$

If $\mu \equiv 1$, then mass = length of curve $\int_{t_0}^{t_1} |g'(t)| dt$

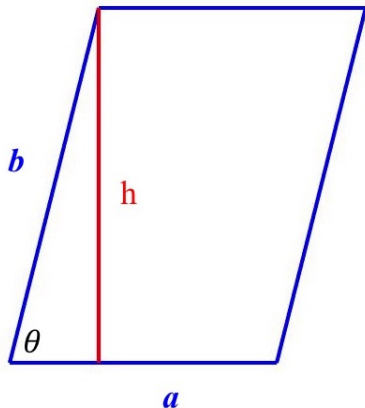
Generalize To Surfaces

Let D be region in plane and $g : D \rightarrow \mathbb{R}^3$ with $g(u, v) = (g_1, g_2, g_3)$ where each component function g_i is continuously differentiable.

There are two natural tangent vectors: $g_u = \frac{\partial g}{\partial u}$ and $g_v = \frac{\partial g}{\partial v}$,
These determine a tangent plane.

S is a **Smooth Surface** if these two vectors are linearly independent.

Note that $\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}$ is normal to the plane with
 $|\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}| = |\frac{\partial g}{\partial u}| |\frac{\partial g}{\partial v}| \sin \theta$
= Area of Parallelogram Spanned by the Vectors



$$\sin \theta = \frac{h}{|\mathbf{b}|} \text{ so } h = |\mathbf{b}| \sin \theta$$

$$\text{Area of Parallelogram} = (\text{Base})(\text{Height}) = |\mathbf{a}||\mathbf{b}| \sin \theta$$

$$\mathbf{a} = g_u, \mathbf{b} = g_v$$

$$|g_u \times g_v| = |g_u||g_v| \sin \theta$$

Surface Area

$$\sigma(S) = \iint_D \left| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right| dudv = \iint_D |g_u \times g_v| dudv$$

If $\mu(g(u, v))$ is density, then mass =

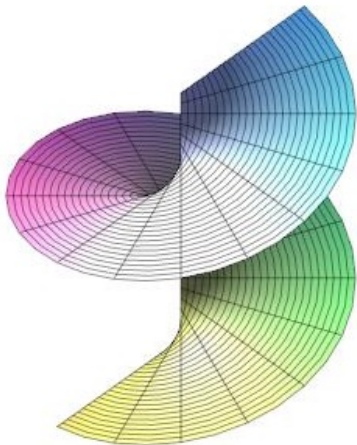
$$\iint_D \mu d\sigma = \iint_D \mu(g(u, v)) |g_u \times g_v| dudv$$

Plotting Parametrized Surface in *Maple*:

`plot3d([g1(u, v), g2(u, v), g3(u, v)], u = ..., v = ...)`

Area of a Spiral Ramp

$$g(u, v) = (u \cos v, u \sin v, v), 0 \leq u \leq 1, 0 \leq v \leq 3\pi$$



Area of a Spiral Ramp

$$g(u, v) = (u \cos v, u \sin v, v), 0 \leq u \leq 1, 0 \leq v \leq 3\pi$$

$$g_u = (\cos v, \sin v, 0), g_v = (-u \sin v, u \cos v, 1)$$

$$\begin{aligned} g_u \times g_v &= \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 1 \end{vmatrix} \\ &= \left(\begin{vmatrix} \sin v & 0 \\ u \cos v & 1 \end{vmatrix}, - \begin{vmatrix} \cos v & 0 \\ -u \sin v & 1 \end{vmatrix}, \begin{vmatrix} \cos v & \sin v \\ -u \sin v & u \cos v \end{vmatrix} \right) \\ &= (\sin v, -\cos v, u) \end{aligned}$$

$$\text{Then } |g_u \times g_v| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1 + u^2}$$

$$\text{Area} = \int_{v=0}^{v=3\pi} \int_{u=0}^1 \sqrt{1 + u^2} \, du \, dv$$

If density is $\mu(\mathbf{x}) = u$, then

Mass =

$$\begin{aligned} \int_{v=0}^{v=3\pi} \int_{u=0}^1 u(1 + u^2)^{1/2} \, du \, dv &= \int_{v=0}^{v=3\pi} \left[\frac{1}{3}(1 + u^2)^{3/2} \right]_0^1 \, dv \\ &= \int_{v=0}^{v=3\pi} \frac{1}{3} [2^{3/2} - 1^{3/2}] \, dv = 3\pi \frac{1}{3} [2^{3/2} - 1] = \pi [2^{3/2} - 1] \end{aligned}$$