

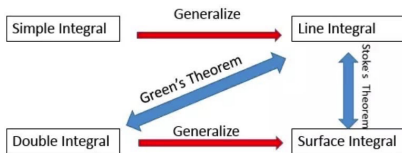
Last Day of Classes!



MATH 224: Vector Calculus

MULTIPLE INTEGRALS

Stokes Theorem



Class 36: Monday May 11, 2026



Notes on Assignment 32
MATH 224 Sticker
History of Stokes' Theorem

Announcements

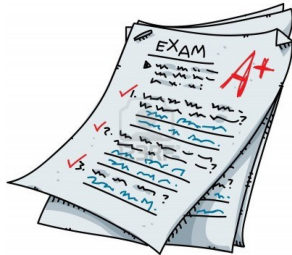
- ▶ **Course Response Forms Today**

- ▶ Link: <https://crfaccess.middlebury.edu/student/> or **go/crf**
- ▶ Available Until 11:05 AM

- ▶ **Additional Office Hours This Week**

Wednesday: 9 – 11 AM

Final Exam



Friday: 9 AM – Noon:
Warner 011 (Here)
One Sheet of Notes

Today:

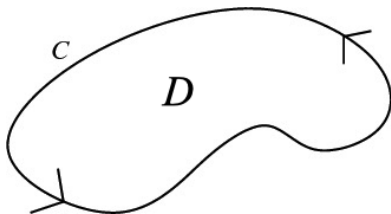
Stokes' Theorem

$$\int_S \operatorname{curl} \mathbf{F} = \int_{\partial S} \mathbf{F}$$

S is a Surface in \mathbb{R}^3

Vector Field Theorems
Plane

$$\mathbf{F} : \mathcal{R}^2 \rightarrow \mathcal{R}^2$$



D is 2-dimensional

$C = \partial D$ is 1-dimensional

Green – Ostrogradski

$$\int_D \text{curl } \mathbf{F} = \int_C \mathbf{F}$$

Gauss (Divergence)

$$\int_D \text{div } \mathbf{F} = \int_C \mathbf{F} \cdot \mathbf{N}$$

Positive Orientation

Setting: Let D be a plane region bounded by a curve traced out in a counterclockwise direction by some parametrization

$$h : \mathcal{R}^1 \rightarrow \mathcal{R}^2 \text{ for } a \leq t \leq b.$$

Let $S = g(D)$ be the image of D where $g : \mathcal{R}^2 \rightarrow \mathcal{R}^3$ so that S is a 2-dimensional surface in 3-space whose border γ corresponds to the boundary of D .

We say that γ inherits the **positive orientation** with respect to S . The composition $g(h(t))$ describes the border of S . Denote by ∂S the **positively oriented border** of S .

Vector Field in \mathcal{R}^3 : $\mathbf{F}(\mathbf{x}) = (F(\mathbf{x}), G(\mathbf{x}), H(\mathbf{x}))$ where each of F, G, H is a real-valued function of 3 variables.

$$\text{Curl } \mathbf{F}(\mathbf{x}) = (H_y(\mathbf{x}) - G_z(\mathbf{x}), F_z(\mathbf{x}) - H_x(\mathbf{x}), G_x(\mathbf{x}) - F_y(\mathbf{x}))$$

Stokes' Theorem: Let S be a piece of smooth surface in \mathcal{R}^3 , parametrized by a twice continuously differentiable function g . Assume that D , the parameter domain of g , is a finite union of simple regions bounded by a piecewise smooth curve. If \mathbf{F} is a continuously differentiable vector field defined on S , then

$$\int_S \text{Curl } \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{x}$$

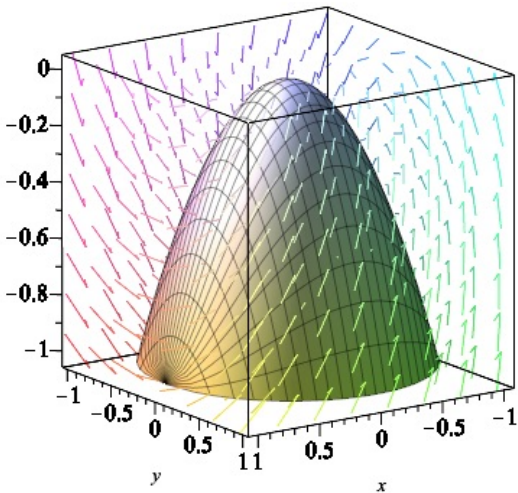
where ∂S is the positively oriented border of S .

[Note: If $\mathbf{F} = (F, G, 0)$ where F and G are independent of z , then Stokes' Theorem reduces to Green's Theorem. Thus Stokes generalizes Green.]

Example: Verify Stokes Theorem where

$$\mathbf{F}(x, y, z) = (z, x, y)$$

$$S : g(u, v) = (u, v, 1 - u^2 - v^2), u^2 + v^2 \leq 1.$$



Example: Verify Stokes Theorem where

$$\mathbf{F}(x, y, z) = (z, x, y)$$

$$S : g(u, v) = (u, v, 1 - u^2 - v^2), u^2 + v^2 \leq 1.$$

$$\text{Parametrize } \partial S \text{ by } (\cos t, \sin t, 0), 0 \leq t \leq 2\pi.$$

Then $g(u, v) = (\cos t, \sin t, 0)$ and $g'(u, v) = (-\sin t, \cos t, 0)$

$$\mathbf{F}(g(u, v)) = (1 - u^2 - v^2, u, v) = (0, \cos t, \sin t)$$

$$\mathbf{F}(g(u, v)) \cdot g'(u, v) = (0, \cos t, \sin t) \cdot (-\sin t, \cos t, 0) = \cos^2 t$$

$$\int_{\partial S} \mathbf{F} = \int_0^{2\pi} \cos^2 t \, dt = \int_0^{2\pi} \frac{1 + \cos 2t}{2} \, dt = \frac{1}{2} \left[t + \frac{\sin 2t}{2} \right]_0^{2\pi} = \pi$$

$$\text{Now } \int_S \text{curl } \mathbf{F} = \int_S \text{curl } (z, x, y)$$

$$\text{curl } \mathbf{F} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{pmatrix} = (1 - 0, -(0 - 1), 1 - 0) = (1, 1, 1)$$

Thus we want to integrate $(1, 1, 1)$ over S .

$$\text{Here } g(u, v) = (u, v, 1 - u^2 - v^2)$$

$$\text{so } g_u = (1, 0, -2u), g_v = (0, 1, -2v)$$

$$\text{and } g_u \times g_v = (2u, 2v, 1) \text{ [work it out]}$$

$$\int_S \text{curl } \mathbf{F} = \iint_D (1, 1, 1) \cdot (2u, 2v, 1) du dv = \iint_D 2u + 2v + 1 du dv$$

which equals (using polar coordinates)

$$\int_{r=0}^{r=1} \int_{\theta=0}^{\theta=2\pi} (2r \cos \theta + 2r \sin \theta + 1) r dr d\theta$$

$$\int_S \operatorname{curl} \mathbf{F} = \iint_D (1, 1, 1) \cdot (2u, 2v, 1) du dv = \iint_D 2u + 2v + 1 du dv$$

which equals (using polar coordinates)

$$\int_{r=0}^{r=1} \int_{\theta=0}^{\theta=2\pi} (2r \cos \theta + 2r \sin \theta + 1) r dr d\theta$$

$$= \int_{r=0}^{r=1} [2r^2 \sin \theta - 2r^2 \cos \theta + r\theta]_{\theta=0}^{\theta=2\pi} dr$$

$$= \int_{r=0}^{r=1} 2\pi r dr = 2\pi \left[\frac{r^2}{2} \right]_{r=0}^{r=1} = \pi$$

Consequences of Stokes' Theorem

$$\int_S \operatorname{curl} \mathbf{F} = \int_{\partial S} \mathbf{F}$$

S is a Surface in \mathbb{R}^3



George Gabriel Stokes
August 13, 1819 – February 1, 1903
[Stokes Biography](#)

Interpretation of Curl

(1) The direction of $\text{curl } \mathbf{F}(\mathbf{x})$ is the axis about which \mathbf{F} rotates most rapidly at \mathbf{x} . The length of $\text{curl } \mathbf{F}(\mathbf{x})$ is the maximum rate of rotation at \mathbf{x} .

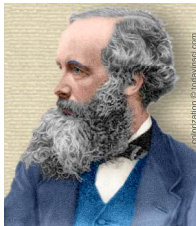
(2) **Maxwell's Equations:** $\text{curl } \mathbf{B} = \mathbf{I}$ where \mathbf{I} is the vector current flow in an electrical conductor and \mathbf{B} is the magnetic field which the current flow induces in the surrounding space.

Stokes' Theorem then yields **Ampere's Law:**

$$\int_S \mathbf{I} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{B} \cdot d\mathbf{x},$$

the total current flux across S is the circulation of the magnetic field around the border curve ∂S that encircles the conductor.

Definitions: A vector field \mathbf{F} is **divergent-free** if $\text{div } \mathbf{F} = 0$ and \mathbf{F} is **curl-free** if $\text{curl } \mathbf{F} = \mathbf{0}$.



James Clerk Maxwell (June 13, 1831 – November 5, 1879)

Maxwell Biography

Name	Equation	
	Integral form	Differential form
Faraday's law of induction	$\oint_c \vec{E} \cdot d\vec{l} = -\iint_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$	$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
Ampère-Maxwell law	$\oint_c \vec{H} \cdot d\vec{l} = \iint_s \vec{J} \cdot d\vec{S} + \iint_s \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S}$	$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$
Gauss' electric law	$\oiint_s \vec{D} \cdot d\vec{S} = \iiint_V \rho dV$	$\nabla \cdot \vec{D} = \rho$
Gauss' magnetic law	$\oiint_s \vec{B} \cdot d\vec{S} = 0$	$\nabla \cdot \vec{B} = 0$

