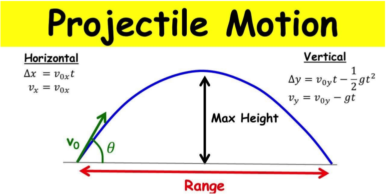
MATH 224: Vector Calculus



February 16, 2026

Handouts

Notes on Assignment 3

Assignment 4

MATLAB Basic Functions Reference

MATLAB Practice Exercises for Vector Calculus Students

MATLAB Workshops

Today: 3:30 – 5 PM

Tomorrow: 3:30 — 5PM

Room 168 inside Q-Center at BiHall

Review: Curves and Tangent Lines

Example 1: $\mathbf{F}(x) = (x^3 + 7x + 3, 8 + \sin x), -2 \leq x \leq 2.$

$$\mathbf{F}(0) = (3, 8)$$

$$\text{So } \mathbf{F'}(x) = (3x^2 + 7, \cos x)$$

$$\text{implying } \mathbf{F'}(0) = (7, 1)$$

Tangent Line

$$\mathbf{L}(\mathrm{t}) = \mathbf{F}(0) + \mathrm{t}\ \mathbf{F'}(0)$$

$$= (3,8) + (7,1)$$

$$= (3 + 7\mathrm{t},\ 8 + \mathrm{t})$$

Review: Curves and Tangent Lines

Example 2: $\mathbf{F}(x) = (x^3 + 7x + 3, 8 + \sin x, \ln 1 + x^2), -2 \leq x \leq 2.$

$$\mathbf{F}(0) = (3, 8, 0)$$

So $\mathbf{F'}(x) = (3x^2 + 7, \cos x, \frac{2x}{1+x^2})$

implying $\mathbf{F'}(0) = (7, 1, 0)$

Tangent Line

$$\mathbf{L}(\text{t}) = \mathbf{F}(0) + \text{t}\mathbf{F}'(0)$$

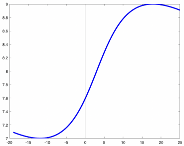
$$= (3{,}8{,}0) + (7{,}1{,}0)$$

$$= (3 + 7\text{t},\ 8 + \text{t},\ 0)$$

Plotting Vector-Valued Functions in MATLAB

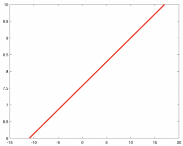
Curve: $\mathbf{F}(x) = (x^3 + 7x + 3, 8 + \sin x), -2 \leq x \leq 2$

```
x = linspace(-2, 2, 100);
plot( x.^3 + 7*x +3, 8 + sin(x), 'b', 'LineWidth', 3)
```



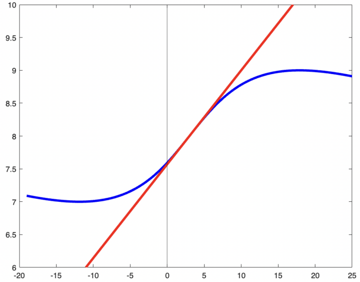
Tangent Line: $\mathbf{L}(x) = (3 + 7x, 8 + x)$

```
x = [-2 : .02 : 2];
plot(3 + 7*x, 8 + x, 'r', 'LineWidth',3)
```



Plot Curve and Tangent Line Together

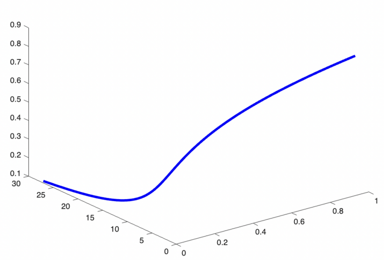
```
hold on
x = [-2 : .02 : 2];
plot( x.^3 + 7*x +3, 8 + sin(x), 'b', 'LineWidth',3)
plot(3 + 7*x, 8 + x, 'r', 'LineWidth',3)
xline(0)
```



Plotting Vector-Valued Functions in MATLAB

$$F(t) = (t^7, t^{-3}, \sin t^2), 1/3 <= t <= 1$$

```
t = [1/3 : .01 : 1];
plot3( t.^7, t.^-3 , sin(t.^2) , 'b', 'LineWidth',3)
```



Integrating Vector–Valued Functions of a Real Number

Example 3: Find all vector–valued functions of a real number $\mathbf{p}(t)$ such that

$$\mathbf{p}'(t) = \left(\frac{1}{t^2+1}, \frac{t}{t^2+1}\right)$$

Solution:

$$\mathbf{p}(t) = \int \mathbf{p}'(t)dt = \int \left(\frac{1}{t^2+1}, \frac{t}{t^2+1}\right) dt$$

$$= \left(\arctan t, \frac{1}{2}\ln\left(t^2+1\right)\right) + (C_1, C_2)$$

$$= \left(\arctan t + C_1, \frac{1}{2}\ln\left(t^2+1\right) + C_2\right)$$

Example 3 Continued: $\mathbf{p}(t) = (\arctan t + C_1, \frac{1}{2}\ln(t^2+1) + C_2)$

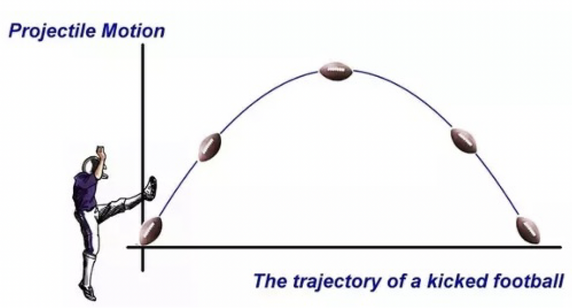
(i) Find the particular solution so that $\mathbf{p}(0) = (3, 4)$

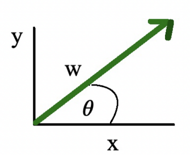
$$3 = \arctan 0 + C_1 = 0 + C_1 \text{ so } C_1 = 3$$
$$4 = \frac{1}{2}\ln(0^2+1) + C_2 = 0 + C_2 \text{ so } C_2 = 4$$
$$\text{Hence } \mathbf{p}(t) = (\arctan t + 3, \frac{1}{2}\ln(t^2+1) + 4)$$

(ii) Find the particular solution so that $\mathbf{p}(1) = (a, b)$

$$a = \arctan 1 + C_1 = \pi/4 + C_1 \text{ so } C_1 = a - \pi/4$$
$$b = \frac{1}{2}\ln(t^2+1) + C_2 = \frac{1}{2}\ln 2 + C_2 \text{ so } C_2 = a - \pi/4 - \frac{1}{2}\ln 2$$
$$\text{Hence } \mathbf{p}(t) = (\arctan t + a - \pi/4, \frac{1}{2}\ln(t^2+1) + a - \pi/4 - \frac{1}{2}\ln 2)$$

Projectile Motion


Projectile Motion
The trajectory of a kicked football



Initial Velocities: $x_0' = w\cos\theta, y_0' = w\sin\theta$

x and y are function of time t.

$$x'' = 0$$
$$x' = C = w\cos\theta$$
$$x = w\cos\theta t + x_0$$

$$y'' = -g$$
$$y' = -gt + C = -gt + w\sin\theta$$
$$y = -\frac{g}{2}t^2 + (w\sin\theta)t + y_0$$

$$x(t) = w\cos\theta t + x_0, y(t) = -\frac{g}{2}t^2 + (w\sin\theta)t + y_0$$

$$x(t) = w\cos\theta t + x_0$$
$$y(t) = -\frac{g}{2}t^2 + (w\sin\theta)t + y_0$$

Suppose $x_0 = 0, y_0 = 0$

Then

$$x(t) = w\cos\theta t$$
$$y(t) = -\frac{g}{2}t^2 + (w\sin\theta)t$$

Note:

$$t = \frac{x}{w\cos\theta}$$
$$y = -\frac{g}{2}\left(\frac{x^2}{w^2\cos^2\theta}\right) + \frac{w\sin\theta}{w\cos\theta}x$$

Graph of y versus x is a Downward Pointing Parabola

$$x(t) = w\cos\theta\, t$$
$$y(t) = -\frac{g}{2}t^2 + (w\sin\theta)t = t\left(-\frac{g}{2}t + w\sin\theta\right)$$

Now $y = 0$ at $t = 0$ and when $w\sin\theta = \frac{g}{2}t$

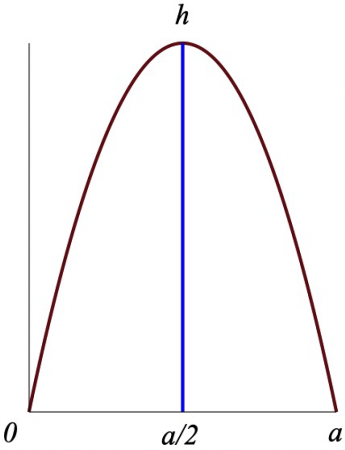
which occurs when $t = \frac{w\sin\theta}{g}$

At this time

$$x = w\cos\theta\left(\frac{w\sin\theta}{g}\right) = \frac{2w^2\sin\theta\cos\theta}{g} = \frac{w^2}{g}\sin 2\theta$$

So Maximum Horizontal Distance occurs when $\sin(2\theta) = 1$;

that is, $\theta = \frac{\pi}{4}$



Maximum Height h

$$\frac{a}{2} = w\cos\theta,\ h = -\frac{g}{2}t^2 + w\sin\theta t$$

Maximuum h occurs when $t = \frac{w\sin\theta}{g}$ where $h'(t) = 0$.

At this time, $x = w\cos\theta\frac{w\sin\theta}{g} = \frac{w^2}{g}\sin\theta\cos\theta$

$$h = -\frac{g}{2}t^2 + w\sin\theta t$$
$$\text{At } t = \frac{w\sin\theta}{g}$$
$$h = -\frac{g}{2}\left(\frac{w\sin\theta}{g}\right)^2 + w\sin\theta\left(\frac{w\sin\theta}{g}\right)$$
$$= -\frac{g}{2}\frac{w^2\sin^2\theta}{g^2} + \frac{w^2\sin^2\theta}{g}$$
$$= -\frac{w^2 sin^2\theta}{2g} + \frac{w^2\sin^2\theta}{g}$$
$$= \frac{w^2\sin^2\theta}{2g}$$
$$\text{where } x = \frac{w^2}{g}\sin\theta\cos\theta$$

To have $y = h$ at $x = a/2$:

$$h = \frac{w^2 \sin^2\theta}{2g}, \frac{a}{2} = \frac{w^2}{g} \sin\theta \cos\theta$$

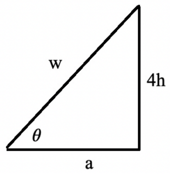
$$\frac{h}{a/2} = \left(\frac{w^2 \sin^2\theta}{2g}\right)\left(\frac{g}{w^2 \sin\theta \cos\theta}\right)$$

$$\frac{2h}{a} = \frac{\sin\theta}{2\cos\theta}$$

$$\frac{4h}{a} = \tan\theta$$

$$\theta = \arctan\left(\frac{4h}{a}\right)$$

Recall $h = \frac{w^2 \sin^2 \theta}{2g}$ and $\tan \theta = \frac{4h}{a}$



$w = \sqrt{16h^2 + a^2}$ so $\sin \theta = \frac{4h}{\sqrt{16h^2+a^2}}$

$h = \frac{w^2}{2g} = \frac{w^2}{2g}\left(\frac{16h^2}{16h^2+a^2}\right)$ Solve for w^2:

$$w^2 = \frac{(2gh)(16h^2+a^2)}{16h^2} = \frac{g(16h^2+a^2)}{8h}$$

Real-Valued Functions of Vectors

Example: $f : \mathbb{R}^2 \to \mathbb{R}^1$

$$f(x,y) = \begin{cases} \frac{xy}{x^2+2y^2} & (x,y) \neq (0,0) \\ 0 & (0,0) \end{cases}$$

What happens as (x, y) approaches (0,0)?
Consider approaching along line $y = mx$. Then

$$\frac{xy}{x^2 + 2y^2} = \frac{xmx}{x^2 + 2(m^2x^2)} = \frac{m}{1 + 2m^2}$$