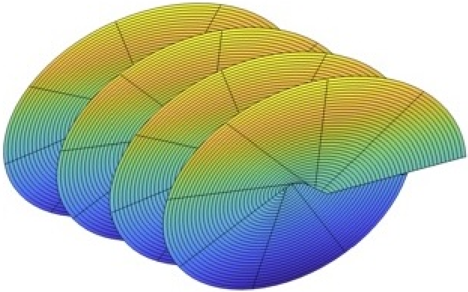
MATH 224: Vector Calculus



Class 8: February 25, 2026

- Notes on Assignment 7
- Assignment 8
- Unified Treatment Of Tangent Lines and Planes

Example from last time: $f(x,y) = x^2y$

$$f_x(x,y) = 2xy \text{ and } f_y(x,y) = x^2$$

$$\text{at } \mathbf{a} = (p,q) : f_x(p,q) = 2pq \text{ and } f_y(p,q) = p^2$$

Can also write as $f_x(\mathbf{a})2pq, f_y(\mathbf{a}) = p^2$

gradient vector at $(p,q) = \nabla f(\mathbf{a}) = \nabla f(p,q) = \left(2pq, p^2\right)$

Can also write as $\nabla f(\mathbf{a}) = (f_x(\mathbf{a}), f_y(\mathbf{a}))$

Now let $\mathbf{w} = (w_1, w_2)$

Want $f_{\mathbf{w}}(p,q) = f_{\mathbf{w}}(\mathbf{a})$

$$f_{\mathbf{w}}(p,q) = \lim_{t\to 0} \frac{f(p+tw_1, q+tw_2) - f(p,q)}{t}$$

$$f_{\mathbf{w}}(p,q) = f_{\mathbf{w}}(\mathbf{a}) = \lim_{t\to 0} \frac{f(\mathbf{a}+t\mathbf{w}) - f(\mathbf{a})}{t}$$

$$f_{\mathbf{w}}(p,q) = \lim_{t\to 0} \frac{f(p+tw_1, q+tw_2) - f(p,q)}{t}$$

$$f_{\mathbf{w}}(p,q) = \lim_{t\to 0} \frac{(p+tw_1)^2(q+tw_2) - p^2q}{t}$$

$$f_{\mathbf{w}}(p,q) = \lim_{t\to 0} \frac{(p^2+2ptw_1+t^2{w_1}^2)(q+tw_2) - p^2q}{t}$$

$$= \lim_{t\to 0} \frac{(p^2q+2ptw_1q+t^2w_1^2q+p^2tw_2+2ptw_1tw_2+t^2w_1^2tw_2) - p^2q}{t}$$

$$= \lim_{t\to 0} \frac{2ptw_1q+t^2w_1^2q+p^2tw_2+2pt^2w_1w_2+t^3w_1^2w_2)}{t}$$

$$f_{\mathbf{w}}(p,q) =$$

$$= \lim_{t\to 0} \frac{2ptw_1q + t^2w_1^2q + p^2tw_2 + 2pt^2w_1w_2 + t^3w_1^2w_2)}{t}$$

$$= \lim_{t\to 0} \frac{(t)2pw_1q + tw_1^2q + p^2w_2 + 2ptw_1w_2 + t^2w_1^2w_2)}{t}$$

$$= \lim_{t\to 0}(2pw_1q + tw_1^2q + p^2w_2 + 2ptw_1w_2 + t^2w_1^2w_2)$$

$$= 2pqw_1 + p^2w_2 = (2pq, p^2)\cdot(w_1, w_2)$$

$$= \nabla f(p,q)\cdot\mathbf{w} = \nabla f(\mathbf{a})\cdot\mathbf{w}$$

Announcements

Exam 1: Next Monday, 7 PM -
No Time Limit

No Books,Computers, Smart Phones, etc.

One Page of Your Own Notes OK

Tangent Plane To Graph of $f : \mathbb{R}^n \to \mathbb{R}^1$ at point $(\mathbf{a}, f(\mathbf{a}))$

$$n = 2 : \; T(\mathbf{x}) = f(\mathbf{a}) + (f_x(\mathbf{a}), f_y(\mathbf{a})) \cdot (\mathbf{x} - \mathbf{a})$$

In general,

$$T(\mathbf{x}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})$$

where $\nabla f(\mathbf{a}) = (f_1)(\mathbf{a}), f_2(\mathbf{a}, ..., f_n(\mathbf{a})$

	Tangent Hyperplane
$n = 1$	Ordinary Tangent Line
$n = 2$	Tangent Plane

Example: $f(x,y,z) = \frac{x^2 y}{z}$

Note: $f : \mathbb{R}^3 \to \mathbb{R}^1$ so GRAPH lives in $\mathbb{R}^4$.

Find Equation of Tangent Hyperplane at $\mathbf{a} = (-3, 4, 2)$

$$f_x(x,y,z) = \tfrac{2xy}{z}$$

$$f_y(x,y,z) = \tfrac{x^2}{z} \; so \nabla f(x,y,z) = \left(\frac{2xy}{z}, \frac{x^2}{z}, -\frac{x^y}{z^2} \right)$$

$$f_z(x,y,z) = -\tfrac{x^y}{z^2}$$

at $\mathbf{a} = (-3,4,2)$: $f(\mathbf{a}) = \frac{(-3)^2 \times 4}{2} = 18$

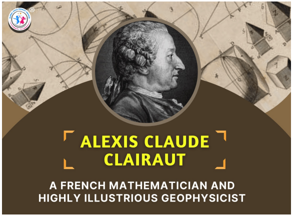
$$\nabla f(\mathbf{a}) = \left(\frac{(2)(-3)(4)}{2}, \frac{(-3)^2}{2}, \frac{-(-3)^2(4)}{2} \right) = \left(-12, \frac{9}{2}, -9 \right)$$

Equation of Tangent Hyperplane is

$$w = 18 + \left(-12, \frac{9}{2}, -9 \right) \cdot (x+3, y-4, z-2)$$

Clairaut's Theorem on Equality of Mixed Partials

If f_{xy} and f_{yx} are continuous at $\mathbf{a}$, then $f_{xy}(\mathbf{a}) = f_{yx}(\mathbf{a})$



May 7, 1713 – May 17, 1765

Proof: End of Section 3.3 in text.

Mean Value Theorem: Suppose f is a real-valued function of a real variable. If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there is at least one number c in the open interval such that $(b - a)f'(c) = f(b) - f(a)$.

Clairaut's Theorem on Equality of Mixed Partials

If f_{xy} and f_{yx} are continuous at $\mathbf{a}$, then $f_{xy}(\mathbf{a}) = f_{yx}(\mathbf{a})$

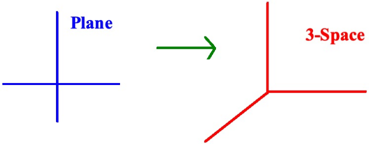
$$f(x,y) = \left\{ \begin{matrix} 2xy\frac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{matrix} \right\}$$

It Turns Out That

$$f_{xy}(0,0) = -2$$
$$f_{yx}(0,0) = +2$$

Mixed Partials Are Not Equal

Parameterized Surfaces



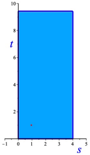
Function from $\mathbb{R}^2 \to \mathbb{R}^3$

Domain:	Patch in Plane
Image:	Surface in Space
Graph:	Lives in $\mathbb{R}^5$

Need for Parameterizations: Graph of $f : \mathbb{R}^1 \to \mathbb{R}^1$ is a curve but not every curve is the graph of such a function

Similarly, graph of $f : \mathbb{R}^2 \to \mathbb{R}^1$ is a surface but not every surface is the graph of such a function.

Example: $\sigma(s,t) = (s\cos t, s\sin t, t), 0 \le s \le 4, 0 \le t \le 3\pi$

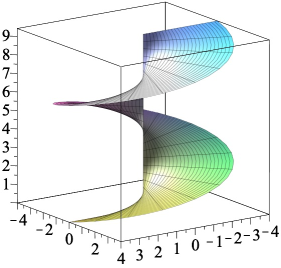


$$\text{Point: } (1, \pi/4) \text{ so } \sigma(1,\pi/4) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)$$

$$\sigma_s(s,t) = (\cos t, \sin t, 0) \text{ and } \sigma_t(s,t) = (-s\sin t, s\cos t, 1)$$

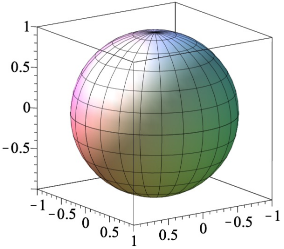
At $\left(1, \frac{\pi}{4}\right)$, representation of the tangent plane is

$$\sigma\left(1, \frac{\pi}{4}\right) + \sigma_s\left(1, \frac{\pi}{4}\right) s + \sigma_t\left(1, \frac{\pi}{4}\right) t$$



Parameterize Unit Sphere

$$\sigma(s,t) = (\cos t \cos s, \sin t \cos s, \sin s), 0 \le s \le 2\pi, 0 \le t \le 2\pi$$

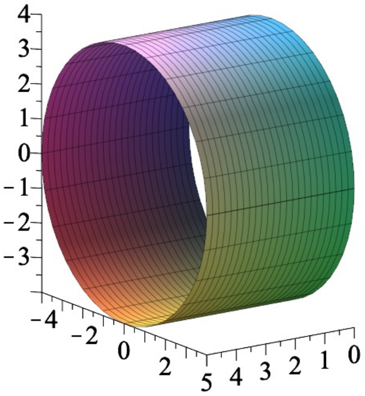


$$x = \cos t \cos s, y = \sin t \cos s, z = \sin s$$

$$\begin{aligned} x^2 + y^2 + z^2 &= \cos^2 t \cos^2 s + \sin^2 t \cos^2 s + \sin^2 s \\ &= \cos^2 s(\cos^2 t + \sin^2 t) + \sin^2 s \\ &= \cos^2 s + \sin^2 s = 1 \end{aligned}$$

Parameterize Cylinder

$$x = s, y = 4\cos t, z = 4\sin t, 0 \leq s \leq 3, 0 \leq t \leq 2\pi$$



A Note on Notation: We have been writing two and three dimensional vector **horizontally** (x, y, z) rather than vertically $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to conserve space on paper or blackboard, but the vertical form is technically more proper and will make some of the subsequent theorems easier to understand.

See Case 2 $f : \mathbb{R}^2 \to \mathbb{R}^3$ in *Unified Treatment of Tangent Lines and Tangent Lines*.