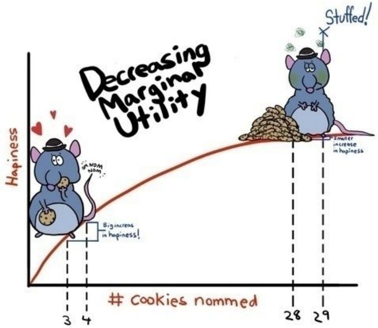
MATH 224: Vector Calculus



Class 9

Friday, February 27, 2026

- Notes on Assignment 8
- Assignment 9

Announcements

Exam 1: Next Monday, 7 PM -
No Time Limit

Last Name A to M: Warner 104
Last Name N to Z: Warner 105

No Books, Computers, Smartphones, etc.

One Page of Notes OK

Focus on Chapters 2 and 3

Tangent Plane To Graph of $f : \mathbb{R}^n \to \mathbb{R}^1$ at point $(\mathbf{a}, f(\mathbf{a}))$

$$n = 2 : T(\mathbf{x}) = f(\mathbf{a}) + (f_x(\mathbf{a}), f_y(\mathbf{a})) \cdot (\mathbf{x} - \mathbf{a})$$

In general,

$$T(\mathbf{x}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})$$

where $\nabla f(\mathbf{a}) = (f_1)(\mathbf{a}), f_2(\mathbf{a}, ..., f_n(\mathbf{a})$

Tangent Hyperplane

$n = 1$ Ordinary Tangent Line

$n = 2$ Tangent Plane

Example: $f(x,y) = x^2 y$

Note: $f : \mathbb{R}^2 \to \mathbb{R}^1$ so GRAPH lives in $\mathbb{R}^3$.

Find Equation of Tangent Hyperplane at $\mathbf{a} = (3,2)$

At $\mathbf{a} = (3,2) : f(\mathbf{a}) = 3^2 \times 2 = 18$

$f_x(x,y) = 2xy$ and $f_y(x,y) = x^2$ so $\nabla f(x,y) = \left(2xy, x^2\right)$

$$\nabla f(\mathbf{a}) = \nabla f(3,2) = (12,9)$$

Equation of Tangent Hyperplane is

$$w = 18 + (12,9) \cdot (x-3, y-2)$$

which we can write in **parameterized form** as

$$w = 18 + (12,9) \cdot (s,t)$$

Example: $f(x,y,z) = \frac{x^2y}{z}$

Note: $f : \mathbb{R}^3 \to \mathbb{R}^1$ so GRAPH lives in $\mathbb{R}^4$.

Find Equation of Tangent Hyperplane at $\mathbf{a} = (-3, 4, 2)$

$$f_x(x,y,z) = \frac{2xy}{z}, f_y(x,y,z) = \frac{x^2}{z}, f_z(x,y,z) = -\frac{x^y}{z^2}$$

$$\nabla f(x,y,z) = \left(\frac{2xy}{z}, \frac{x^2}{z}, -\frac{x^2y}{z^2} \right)$$

at $\mathbf{a} = (-3, 4, 2)$: $f(\mathbf{a}) = \frac{(-3)^2 \times 4}{2} = 18$

$$\nabla f(\mathbf{a}) = \left(\frac{(2)(-3)(4)}{2}, \frac{(-3)^2}{2}, \frac{-(-3)^2(4)}{2} \right) = \left(-12, \frac{9}{2}, -9 \right)$$

Equation of Tangent Hyperplane is

$$w = 18 + \left(-12, \frac{9}{2}, -9 \right) \cdot (x + 3, y - 4, z - 2)$$

$$w = 18 + \left(-12, \frac{9}{2}, -9 \right) \cdot (r, s, t)$$

Tangent Planes To Surfaces

(I) $f : \mathbb{R}^2 \to \mathbb{R}^1$**, a in** $\mathbb{R}^2$

Tangent plane to graph of f at $(\mathbf{a}, f(\mathbf{a}))$:

$$T(\mathbf{x}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})$$

(II): $f : \mathbb{R}^2 \to \mathbb{R}^3$

$$\sigma(s,t) = (f(s,t), g(s,t), h(s,t))$$

$$\sigma_s(s,t) = (f_s, g_s, h_s) \text{ and. } \sigma_t(s,t) = (f_t, g_t, h_t)$$

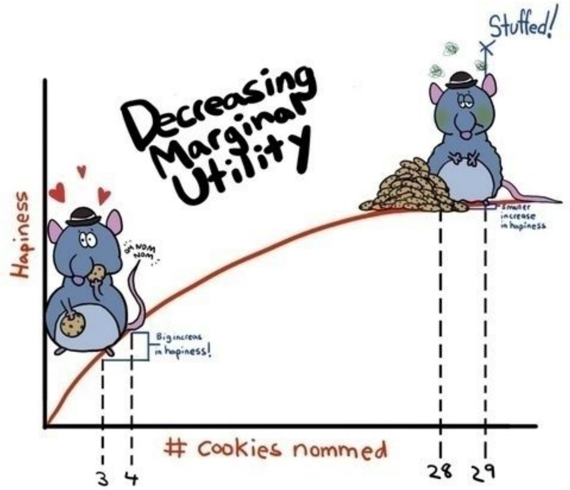
Tangent Plane at $\sigma(\mathbf{a})$:

$$\sigma(\mathbf{a}) + (s,t)\begin{pmatrix} f_s(\mathbf{a}) & g_s(\mathbf{a}) & h_s(\mathbf{a}) \\ f_t(\mathbf{a}) & g_t(\mathbf{a}) & h_t(\mathbf{a}) \end{pmatrix}$$

Note: $1 \times 3 + (1 \times 2)(2 \times 3)$

Writing vectors vertically: $\sigma = \begin{pmatrix} f \\ g \\ h \end{pmatrix}, \sigma' = \begin{pmatrix} f' \\ g' \\ h' \end{pmatrix}$

Tangent Plane: $T\begin{pmatrix} s \\ t \end{pmatrix} = \sigma(\mathbf{a}) + \sigma'(\mathbf{a})\begin{pmatrix} s \\ t \end{pmatrix}$


Decreasing Marginal Utility
Stuffed!
Hapiness
cookies nommed
Big increas in hapiness!
smaller increase in hapiness
3
4
28
29

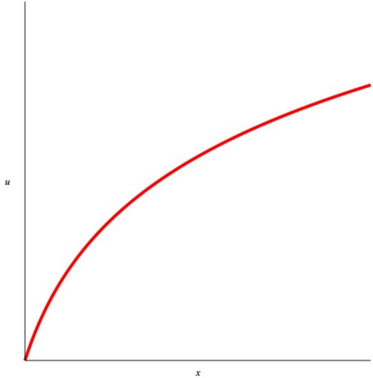
Utility

Utility = happiness, satisfaction, pleasure, usefulness

$u(x), x \geq 0$

Typical Assumptions: u is increasing, concave down function

(”decreasing returns to scale”)



Example: $u(x) = x^{1/3}$ so $u'(x) = \frac{1}{3x^{2/3}}, u''(x) = -\frac{2}{9}x^{-5/3}$

Example: Two Goods with $u(x, y) = \sqrt[3]{xy}$
Each unit of x costs \$35 and each unit of y costs \$80
We have \$D to spend: Budget Constraint: $35x + 80y = D$
Goal: Maximize Utility:

$$80y = D - 35x \text{ so } y = \frac{D - 35x}{80}$$

$$u(x, y) = f(x) = \sqrt[3]{\frac{x(D - 35x)}{80}}$$

f is maximized when $\frac{x(D-35x)}{80}$ is maximized.
$G(x) = x(D - 35x). = Dx - 35x^2.$ has $G'(x) = D - 70x$ and
$G''(xx) = -70$ Hence there is a maximum when $x = D/70$
Then $y = \frac{D-35(D/70)}{80} = D/160$

Clairaut's Theorem on Equality of Mixed Partials

If f_{xy} and f_{yx} are continuous at $\mathbf{a}$, then $f_{xy}(\mathbf{a}) = f_{yx}(\mathbf{a})$

Clairaut's Theorem on Equality of Mixed Partials

If f_{xy} and f_{yx} are continuous at $\mathbf{a}$, then $f_{xy}(\mathbf{a}) = f_{yx}(\mathbf{a})$

$$f(x,y) = \left\{ \begin{matrix} 2xy\frac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{matrix} \right\}$$

It Turns Out That

$$f_{xy}(0,0) = -2$$
$$f_{yx}(0,0) = +2$$

Mixed Partials Are Not Equal