MATH 224: Vector Calculus

Class 28: Wednesday April 22, 2026

Handouts

Notes on Assignment 25
Assignment 26
Weighted Curves

What's Next?

Calculus Along Curves

- Work
- Vector Fields and Line Integrals
- Arc Length and Weighted Curves
- Curvature and Normals
- Flow Lines and Differential Equations

VECTOR FIELDS **F** : $\mathbb{R}^n \to \mathbb{R}^n$

$$\mathbf{F}(\vec{x}) = (F_1(\vec{x}), F_2(\vec{x}), .., F_n(\vec{x}))$$

What is Meaning of $\int_{\mathcal{D}} \mathbf{F}$?

For Now: $\mathcal{D}$ is a one-dimensional set in $\mathbb{R}^n$
$\mathcal{D}$ is a curve defined by a function $g : \mathbb{R}^1 \to \mathbb{R}^n$ on an interval
$a \leq t \leq b$
We denote the **image** of g by γ
Definition The **Line Integral** of **F** over γ is

$$\int_{\gamma} \mathbf{F} \cdot d\vec{x} = \int_a^b \mathbf{F}(g(t)) \cdot g'(t)\, dt$$

Line Integral Example

$$g(t) = (\cos t, \sin t), 0 \leq t \leq 2\pi$$
$$\mathbf{F}(x,y) = (x, yx^2)$$

$$g'(t) = (-\sin t, \cos t)$$
$$\mathbf{F}(g(t)) = \mathbf{F}(\cos t, \sin t) = (\cos t, (\sin t)(\cos^2 t)$$

Value of Line Integral is
$\int_0^{2\pi} \mathsf{F}(g(t)) \cdot g'(t)\, dt = \int_0^{2\pi} -\sin t \cos t + \sin t \cos^2 t \, dt$

Alternative Notation for $n = 2$

$$g(T) = (g_1(t), g_2(t)) = (x(t), y(t))$$

$$\mathbf{F}(x, y) = (F_1(x, y), F_2(x, y))$$

$$\int_\gamma \mathbf{F} \cdot d\vec{x} = int_\gamma(F_1 dx + F_2 dy)$$

In our example, $\int_\gamma (x dx + y x^2 dy)$

Theorem The value of the line integral $\int_\gamma \mathbf{F}$ is independent of the parametrization of γ but in general is dependent on the curve itself.

For some vector fields, the line integral $\int_\gamma \mathbf{F}$ depends only on the **endpoints** of the curve.

Theorem (**The Fundamental Theorem of Calculus for Line Integrals**). Let $f : \mathbb{R}^n \to \mathbb{R}^1$ be continuously differentiable and let $\mathbf{F} = \nabla f$ and suppose $\gamma : \mathbb{R}^1 \to \mathbb{R}^n$ is a continuous curve with endpoints $\vec{a}$ and $\vec{b}$.

Then $\int_\gamma \mathbf{F} = \int_\gamma \nabla f = f(\vec{b}) - f(\vec{a})$.

If **F** $= \nabla f$ for some f, then we call **F**
a **Conservative Vector Field**
or an **Exact Vector Field**

and f is called a **Potential** of **F**

The function $P(\vec{x}) = -f(\vec{x})$ is the **Potential Energy** of the field **F**.

Conservative Vector Field: **F**$(x, y) = (2xy, x^2 + 2y)$
Nonconservative Example **F**$(x, y) = (x, x + 1)$

Application **Conservation of Energy**

$$\mathbf{F}(g(t)) = [m(t)v(t)]' = m'(t)v(t) + m(t)v'(t)$$

(a) $\mathbf{F}(g(t)) \cdot g'(t)$

$$= [m'v + mv'] \cdot g'$$

$$= [m'v + mv']\, v = m'v^2 + mvv'$$

(b) $m(t) =$ Constant implies $m' = 0$

so $\mathbf{F}(g(t)) \cdot g'(t) = mvv'$

$$\int_a^b mvv' \, dt = \frac{mv^2}{2}\bigg|_{t=a}^{t=b}$$

Application **Conservation of Energy**

Suppose **F** is a force field which moves an object of mass m from $\vec{a}$ to $\vec{b}$ along curve γ.

Let g be a parametrization of curve γ and $v(t) = g'(t)$.

Then the work done in moving the object is

$$\frac{1}{2}m|v(t_b)|^2 - \frac{1}{2}m|v(t_a)|^2 \text{(Change in Kinetic Energy)}$$

If **F** is a conservative field, then we can also compute work done by

$\int_\gamma \mathbf{F} = f(\vec{b}) - f(\vec{a}) = p(\vec{a}) - p(\vec{b}) =$ Change in Potential Energy

Equating the two expressions for work, we have

$$\tfrac{1}{2}m|v(t_b)|^2 - \tfrac{1}{2}m|v(t_a)|^2 = p(\vec{a}) - p(\vec{b})$$

$$p(\vec{b}) + \tfrac{1}{2}m|v(t_b)|^2 = p(\vec{a}) + \tfrac{1}{2}m|v(t_a)|^2$$

where $\vec{a}$ and $\vec{b}$ are any 2 points

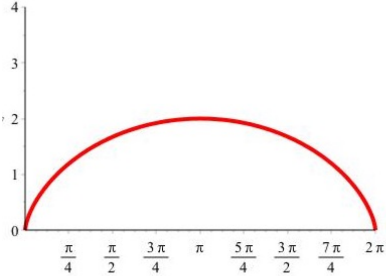
But $T(x)$= sum of Potential and Kinetic Energy

Law of Conservation of Total Energy

Arc Length

Let $g : \mathbb{R}^1 \to \mathbb{R}^n$ be defined on $a \leq t \leq b$. Then the image of g is a curve γ with length $L(\gamma) = \int_a^b |g'(t)|\, dt$.

Example: **Cycloid**: $g(t) = (t - \sin t, 1 - \cos t), 0 \leq t \leq 2\pi$



$$g'(t) = (1 - \cos t, \sin t)$$

$$|g'(t)| = \sqrt{(1 - \cos t)^2 + \sin^2 t} = \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} = \sqrt{2 - 2\cos t} = \sqrt{2(1 - \cos t)} = \sqrt{2(2\sin^2(t/2)} = 2\sin(t/2)$$

$$L(\gamma) = \int_0^{2\pi} 2\sin(t/2)\, dt = -4\cos(t/2)\Big|_0^{2\pi} = 8$$

Other Formulations

$$L(\gamma) = \int_a^b |g'(t)|\, dt$$

If a curve is given by $y = f(x), a \leq x \leq b$, then let $g(t) = (t, f(t))$
$|g'(t)| = |(1, f'(t)| = \sqrt{1 + [f'(t)]^2}$

If $g(t) = (h_1(t), h_2(t)|$, then $|g'(t)| = \sqrt{[h_1']^2 + [h_2']^2}$.

Arc Length Parametrization

Let γ be a curve parametrized by $g(t)$ for $t_0 \leq t \leq t_1$

With $\vec{x}(t) = g(t)$,$\vec{x}$ is position at time t.

Then **arc length function** is $s = s(t) = \int_{t_0}^{t} |g'(t)|\, dt = \int_{t_0}^{t} |x(t)|\, dt$

If $|g'(t)| = 1$ for all t, then we say the **curve is parametrized by arc length**

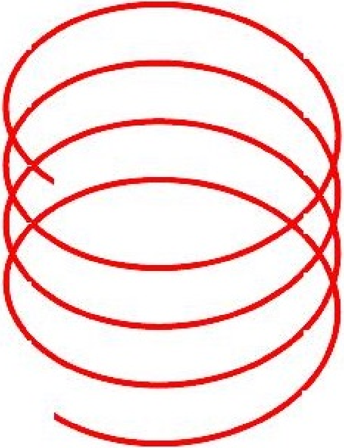
Moving along the curve with uniform speed of 1 means that at time s we are at a point s units along the curve.

Example 1: Unit Circle: $g(t) = (\cos t, \sin t), 0 \leq t \leq 2\pi$

Example 2 Helix: $g(t) = \left(\frac{a\cos t}{\sqrt{a^2+b^2}}, \frac{a\sin t}{\sqrt{a^2+b^2}}, \frac{bt}{\sqrt{a^2+b^2}}\right).$

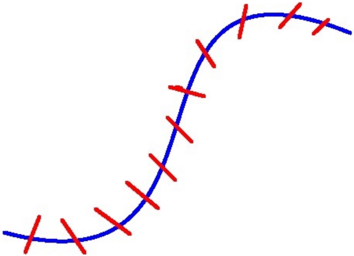
Then $g'(t) = \left(\frac{-a\sin t}{\sqrt{a^2+b^2}}, \frac{a\cos t}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right).$

and $|g'(t)| = \sqrt{\frac{a^2\sin^2 t+a^2\cos^2 t+b^2}{a^2+b^2}} = \sqrt{\frac{a^2+b^2}{a^2+b^2}} = 1$



Mass of a Weighted Curve

Density (μ) is mass per unit length



Total Mass $\sim \sum \mu(point)\times$ Length of short piece of curve

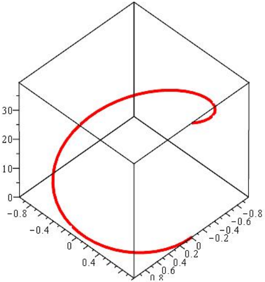
Total **Mass** $= \int \mu(g(t))|g'(t)|\,dt$

Total Mass : $\int \mu(g(t))|g'(t)|\,dt$

Example Spacecurve $g(t) = (\sin t, \cos t, t^2), 0 \leq t \leq 2\pi$

Here $g'(t) = (\cos t, -\sin t, 2t)$

so $|g'(t)| = \sqrt{\cos^2 t + \sin^2 t + 4t^2} = \sqrt{1 + 4t^2}$



Suppose $\mu(x, y, z) = x^2 + y^2 + \sqrt{z} - 1$

Then $\mu(g(t)) = \mu(\sin t, \cos t, t^2) = \cos^2 + \sin^2 t + \sqrt{t^2} - 1$

$$= 1 + t - 1 = t$$

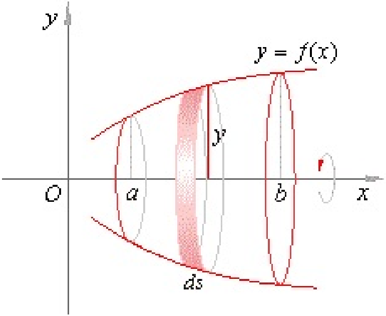
Thus Mass $= \int_0^{2\pi} t\sqrt{1 + 4t^2}\,dt$

$$= \frac{1}{12}(1 + 4t^2)^{3/2}\Big|_0^{2\pi} = \frac{1}{12}\left[(1 + 16\pi^2)^{3/2} - 1\right]$$

Surface of Revolution

S is a surface in $\mathbb{R}^3$ obtained by rotating a plane curve about a straight line in the plane.

Simplest Case: Rotate $y = f(x)$ about x-axis.



$$\text{Volume} = \int_a^b \pi \left[f(x)\right]^2 \; dx$$

$$\text{Surface Area} = \int_a^b 2\pi \sqrt{1 + \left[f(x)\right]^2} \, dx$$

Volume $= \int_a^b \pi \left[f(x)\right]^2 \, dx$

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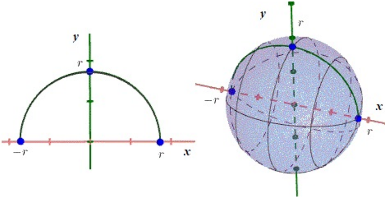
Suppose curve has parametrization $g : \mathbb{R}^1 \to \mathbb{R}^2, t_0 \leq t \leq t_1$

$g(t) = (x(t), y(t))$ with $g(t_0) = (a, f(a))$ and $g(t_1) = (b, f(b))$.

Volume $= \int_{t_0}^{t_1} \pi \left[y(t)\right]^2 x'(t) \, dt$

Surface Area $= \int_{t_0}^{t_1} 2\pi y(t) |g'(t)| \, dt$

Example Revolve Semicircle of radius r about horizontal axis.



$$g(t) = (r\cos t, r\sin t), 0 \le t \le \pi$$

$$\text{Volume } = \int_{t_0}^{t_1} \pi \left[y(t)\right]^2 x'(t)\,dt$$

$$\text{Surface Area } = \int_{t_0}^{t_1} 2\pi y(t)|g'(t)|\,dt$$

$$\text{Surface Area } = \int_{t_0}^{\pi} r^2 2\pi \sin t\,dt$$

$$= -2\pi r^2 \cos t\Big|_0^{\pi} = -2r^2\pi(-1-1) = 4\pi r^2.$$

$$\text{Volume } = \int_0^{\pi} \pi(r\sin t)^2 r\sin t\,dt = \tfrac{4}{3}\pi r^3$$