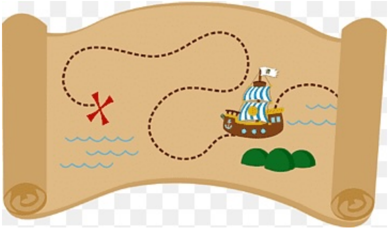
MATH 224: Vector Calculus



Class 29: Friday April 24, 2026

Notes on Assignment 26
Assignment 27

VECTOR FIELDS $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$

$$\mathbf{F}(\vec{x}) = (F_1(\vec{x}), F_2(\vec{x}), .., F_n(\vec{x}))$$

What is Meaning of $\int_{\mathcal{D}} \mathbf{F}$?

For Now: $\mathcal{D}$ is a one-dimensional set in $\mathbb{R}^n$
$\mathcal{D}$ is a curve defined by a function $g : \mathbb{R}^1 \to \mathbb{R}^n$ on an interval
$a \leq t \leq b$
We denote the **image** of g by γ
<u>Definition</u> The **Line Integral** of **F** over γ is

$$\int_{\gamma} \mathbf{F} \cdot d\vec{x} = \int_a^b \mathbf{F}(g(t)) \cdot g'(t)\, dt$$

Theorem The value of the line integral $\int_\gamma \mathbf{F}$ is independent of the parametrization of γ but in general is dependent on the curve itself.

For some vector fields, the line integral $\int_\gamma \mathbf{F}$ depends only on the **endpoints** of the curve.

Theorem (**The Fundamental Theorem of Calculus for Line Integrals**). Let $f : \mathbb{R}^n \to \mathbb{R}^1$ be continuously differentiable and let $\mathbf{F} = \nabla f$ and suppose $\gamma : \mathbb{R}^1 \to \mathbb{R}^n$ is a continuous curve with endpoints $\vec{a}$ and $\vec{b}$.

Then $\int_\gamma \mathbf{F} = \int_\gamma \nabla f = f(\vec{b}) - f(\vec{a})$.

Proof of Fundamental Theorem of Calculus for For Line Integrals:
$\int_\gamma \mathbf{F} = \int_\gamma \nabla f = f(\vec{b}) - f(\vec{a})$ if $\mathbf{F} = \nabla f$ where $f : \mathbb{R}^n \to \mathbb{R}^1$
and $\gamma : \mathbb{R}^1 \to \mathbb{R}^n$ is a continuous curve with endpoints $\vec{a}$ and $\vec{b}$.

Let g be a parameterization of the curve γ so $g : [t_a, t_b] \to \mathbb{R}^n$ with
$g(t_a) = \vec{a}$ and $g(t_b) = \vec{b}$.
Then $\int_\gamma \mathbf{F} = \int_{t_a}^{t_b} F(g(t) \cdot g'(t)\, dt = G(t_b) - G(t_a)$ where G is any antiderivative of $F(g(t) \cdot g'(t)$
But if $G(t) = f(g(t))$, then the Chain Rule says
$G'(t) = \nabla f(g(t)) \cdot g'(t)$ so $f(g(t))$ is one such antiderivative.
Hence the $\int_\gamma \mathbf{F} = f(g(t_b)) - f(g(t_a)) = f(\vec{b}) - f(\vec{a})$

If **F** $= \nabla f$ for some f, then we call **F** a **Conservative Vector Field** or an **Exact Vector Field**

and f is called a **Potential** of **F**

The function $P(\vec{x}) = -f(\vec{x})$ is the **Potential Energy** of the field **F**.

Conservative Vector Field: $\mathbf{F}(x, y) = (2xy, x^2 + 2y)$
Nonconservative Example $\mathbf{F}(x, y) = (x, x + 1)$

Application **Conservation of Energy**

$$\mathbf{F}(g(t)) = [m(t)v(t)]' = m'(t)v(t) + m(t)v'(t)$$

(a) $\mathbf{F}(g(t)) \cdot g'(t)$

$$= [m'v + mv'] \cdot g'$$

$$= [m'v + mv']\, v = m'v^2 + mvv'$$

(b) $m(t) =$ Constant implies $m' = 0$

so $\mathbf{F}(g(t)) \cdot g'(t) = mvv'$

$$\int_a^b mvv'\, dt = \frac{mv^2}{2}\Bigg|_{t=a}^{t=b}$$

Application **Conservation of Energy**

Suppose **F** is a force field which moves an object of mass m from $\vec{a}$ to $\vec{b}$ along curve γ.

Let g be a parametrization of curve γ and $v(t) = g'(t)$.

Then the work done in moving the object is

$$\frac{1}{2}m|v(t_b)|^2 - \frac{1}{2}m|v(t_a)|^2 \text{(Change in Kinetic Energy)}$$

If **F** is a conservative field, then we can also compute work done by $\int_\gamma \mathbf{F} = f(\vec{b}) - f(\vec{a}) = p(\vec{a}) - p(\vec{b}) =$ Change in Potential Energy

Equating the two expressions for work, we have

$\frac{1}{2}m|v(t_b)|^2 - \frac{1}{2}m|v(t_a)|^2 = p(\vec{a}) - p(\vec{b})$

$p(\vec{b}) + \frac{1}{2}m|v(t_b)|^2 = p(\vec{a}) + \frac{1}{2}m|v(t_a)|^2$

where $\vec{a}$ and $\vec{b}$ are any 2 points

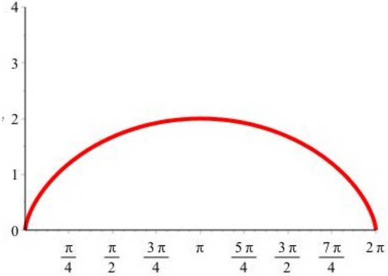
But $T(x)$= sum of Potential and Kinetic Energy

Law of Conservation of Total Energy

Arc Length

Let $g : \mathbb{R}^1 \to \mathbb{R}^n$ be defined on $a \leq t \leq b$. Then the image of g is a curve γ with length $L(\gamma) = \int_a^b |g'(t)|\, dt$.

Example: **Cycloid**: $g(t) = (t - \sin t, 1 - \cos t), 0 \leq t \leq 2\pi$



$$g'(t) = (1 - \cos t, \sin t)$$

$$|g'(t)| = \sqrt{(1 - \cos t)^2 + \sin^2 t} = \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} = \sqrt{2 - 2\cos t} = \sqrt{2(1 - \cos t)} = \sqrt{2(2\sin^2(t/2)} = 2\sin(t/2)$$

$$L(\gamma) = \int_0^{2\pi} 2\sin(t/2)\, dt = -4\cos(t/2)\Big|_0^{2\pi} = 8$$

Other Formulations

$$L(\gamma) = \int_a^b |g'(t)|\, dt$$

If a curve is given by $y = f(x), a \le x \le b$, then let $g(t) = (t, f(t))$
$|g'(t)| = |(1, f'(t)| = \sqrt{1 + [f'(t)]^2}$

If $g(t) = (h_1(t), h_2(t)|$, then $|g'(t)| = \sqrt{[h_1']^2 + [h_2']^2}$.

Arc Length Parametrization

Let γ be a curve parametrized by $g(t)$ for $t_0 \leq t \leq t_1$

With $\vec{x}(t) = g(t)$,$\vec{x}$ is position at time t.

Then **arc length function** is $s = s(t) = \int_{t_0}^{t} |g'(t)|\, dt = \int_{t_0}^{t} |x(t)|\, dt$

If $|g'(t)| = 1$ for all t, then we say the **curve is parametrized by arc length**

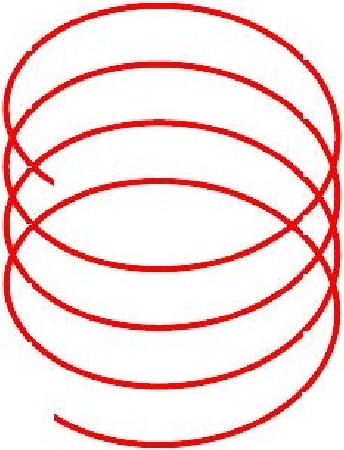
Moving along the curve with uniform speed of 1 means that at time s we are at a point s units along the curve.

Example 1: Unit Circle: $g(t) = (\cos t, \sin t), 0 \leq t \leq 2\pi$

Example 2 Helix: $g(t) = \left(\frac{a\cos t}{\sqrt{a^2+b^2}}, \frac{a\sin t}{\sqrt{a^2+b^2}}, \frac{bt}{\sqrt{a^2+b^2}}\right)$.

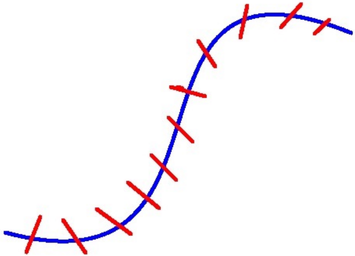
Then $g'(t) = \left(\frac{-a\sin t}{\sqrt{a^2+b^2}}, \frac{a\cos t}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right)$.

and $|g'(t)| = \sqrt{\frac{a^2\sin^2 t+a^2\cos^2 t+b^2}{a^2+b^2}} = \sqrt{\frac{a^2+b^2}{a^2+b^2}} = 1$



Mass of a Weighted Curve

Density (μ) is mass per unit length



Total Mass $\sim \sum \mu(point) \times$ Length of short piece of curve

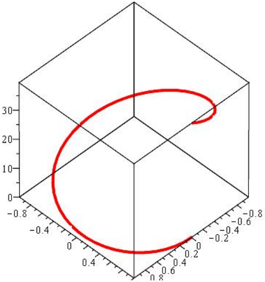
Total **Mass** $= \int \mu(g(t))|g'(t)|\,dt$

Total Mass : $\int \mu(g(t))|g'(t)|\, dt$

Example Spacecurve $g(t) = (\sin t, \cos t, t^2), 0 \le t \le 2\pi$

Here $g'(t) = (\cos t, -\sin t, 2t)$

so $|g'(t)| = \sqrt{\cos^2 t + \sin^2 t + 4t^2} = \sqrt{1 + 4t^2}$



Suppose $\mu(x, y, z) = x^2 + y^2 + \sqrt{z} - 1$

Then $\mu(g(t)) = \mu(\sin t, \cos t, t^2) = \cos^2 + \sin^2 t + \sqrt{t^2} - 1$

$= 1 + t - 1 = t$

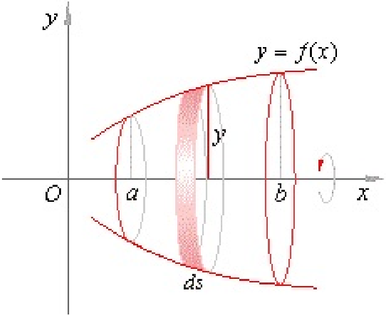
Thus Mass $= \int_0^{2\pi} t\sqrt{1 + 4t^2}\, dt$

$$= \frac{1}{12}(1 + 4t^2)^{3/2}\Big|_0^{2\pi} = \frac{1}{12}\left[(1 + 16\pi^2)^{3/2} - 1\right]$$

Surface of Revolution

S is a surface in $\mathbb{R}^3$ obtained by rotating a plane curve about a straight line in the plane.

Simplest Case: Rotate $y = f(x)$ about x-axis.



$$\text{Volume} = \int_a^b \pi \left[f(x)\right]^2 \; dx$$

$$\text{Surface Area} = \int_a^b 2\pi \sqrt{1 + \left[f(x)\right]^2}\, dx$$

$$\text{Volume} = \int_a^b \pi \left[f(x)\right]^2 \, dx$$

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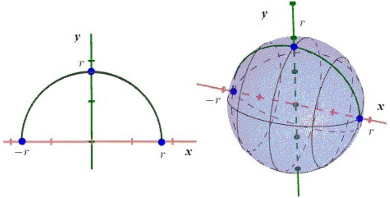
Suppose curve has parametrization $g : \mathbb{R}^1 \to \mathbb{R}^2, t_0 \leq t \leq t_1$

$g(t) = (x(t), y(t))$ with $g(t_0) = (a, f(a))$ and $g(t_1) = (b, f(b))$.

$$\text{Volume} = \int_{t_0}^{t_1} \pi \left[y(t)\right]^2 x'(t) \, dt$$

$$\text{Surface Area} = \int_{t_0}^{t_1} 2\pi y(t) |g'(t)| \, dt$$

Example Revolve Semicircle of radius r about horizontal axis.



$$g(t) = (r\cos t, r\sin t), 0 \le t \le \pi$$
$$\text{Volume } = \int_{t_0}^{t_1} \pi\, [y(t)]^2\, x'(t)\, dt$$
$$\text{Surface Area } = \int_{t_0}^{t_1} 2\pi y(t) |g'(t)|\, dt$$
$$\text{Surface Area } = \int_{t_0}^{\pi} r^2 2\pi \sin t\, dt$$
$$= -2\pi r^2 \cos t \Big|_0^{\pi} = -2r^2\pi(-1-1) = 4\pi r^2.$$
$$\text{Volume } = \int_0^{\pi} \pi (r\sin t)^2 r \sin t\, dt = \frac{4}{3}\pi r^3$$