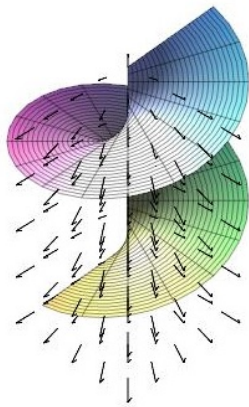


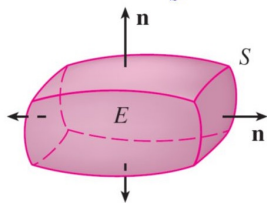
MATH 224: Vector Calculus

Class 35: Friday, May 8, 2026



Divergence Theorem

$$\iiint_E \operatorname{div} \mathbf{F} \, dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$





Notes on Assignment 31
Assignment 32

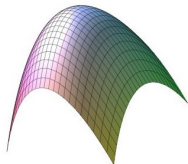
Announcements

Minimizing Travel Costs Project Solutions Due Today

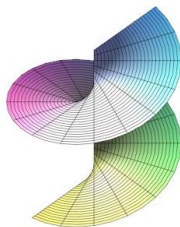
Course Response Forms
In Class Next Monday
Bring Laptop/SmartPhone

Final Exam
Next Friday, May15: 9 AM – Noon

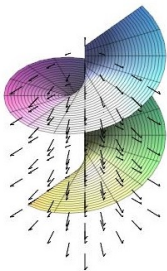
Integrating Vector Fields Over Surfaces



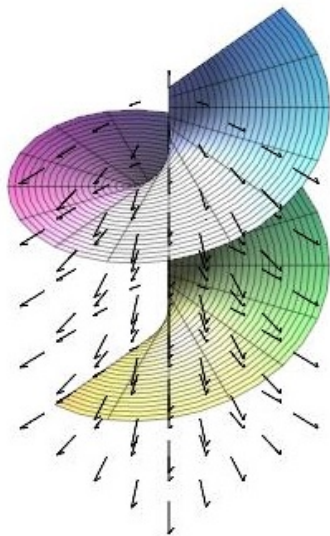
$$g(u, v) = [u, v, -2u^2 - 3v^2]$$



$$g(u, v) = [u \cos v, u \sin v, v]$$



Integrating Vector Fields Over Surfaces



$$g(u, v) = [u \cos v, u \sin v, v]$$

Smooth Curve γ

$$g : I \text{ in } \mathbb{R}^1 \rightarrow \mathbb{R}^n$$

$$\text{Length} = \int_I |g'(t)| dt$$

$$\text{Mass} = \int_I \mu(g(t)) |g'(t)| dt$$

Line Integral

$$\int_{\gamma} \mathbf{F} = \int_I \mathbf{F}(g(t)) \cdot g'(t) dt$$

Smooth Surface S

$$g : D \text{ in } \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

$$\text{Area } \sigma(S) = \iint_D |g_u \times g_v| dudv$$

$$\text{Mass} = \iint_D \mu d\sigma$$

Surface Integral

$$\iint_S \mathbf{F} = \iint_D \mathbf{F}(g(u, v)) \cdot (g_u \times g_v)$$

$$\iint_S \mathbf{F} = \iint_S \mathbf{F} \cdot dS = \iint_S \mathbf{F} \cdot \mathbf{N} d\sigma$$

$\Phi(\mathbf{F}, S) = \iint_S \mathbf{F}$ is **flux** of \mathbf{F} across S .

Surface Integral

Let g be a function from an interval $[t_0, t_1]$ into \mathbb{R}^n with image γ and μ density at $g(t)$.

$$\text{Then Mass of Wire} = \int_{t_0}^{t_1} \mu(t) |g'(t)| dt$$

If $\mu \equiv 1$, then mass = length of curve $\int_{t_0}^{t_1} |g'(t)| dt$

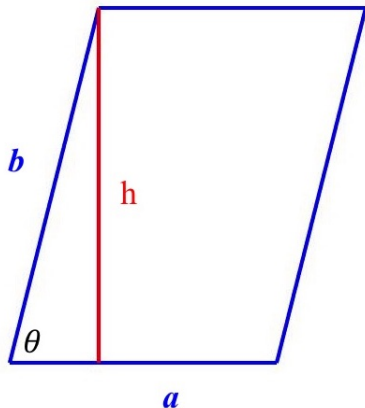
Generalize To Surfaces

Let D be region in plane and $g : D \rightarrow \mathbb{R}^3$ with $g(u, v) = (g_1, g_2, g_3)$ where each component function g_i is continuously differentiable.

There are two natural tangent vectors: $g_u = \frac{\partial g}{\partial u}$ and $g_v = \frac{\partial g}{\partial v}$,
These determine a tangent plane. S is a **Smooth Surface** if these two vectors are linearly independent.

Note that $\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}$ is normal to the plane with
$$\left| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right| = \left| \frac{\partial g}{\partial u} \right| \left| \frac{\partial g}{\partial v} \right| \sin \theta$$

= Area of Parallelogram Spanned by the Vectors



$$\sin \theta = \frac{h}{|\mathbf{b}|} \text{ so } h = |\mathbf{b}| \sin \theta$$

$$\text{Area of Parallelogram} = (\text{Base})(\text{Height}) = |\mathbf{a}||\mathbf{b}| \sin \theta$$

$$\mathbf{a} = g_u, \mathbf{b} = g_v$$

$$|g_u \times g_v| = |g_u||g_v| \sin \theta$$

Surface Area

$$\sigma(S) = \iint_D \left| \frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \right| du dv = \iint_D |g_u \times g_v| du dv$$

If $\mu(g(u, v))$ is density, then mass =

$$\iint_D \mu d\sigma = \iint_D \mu(g(u, v)) |g_u \times g_v| du dv$$

Plotting Parametrized Surface in *MATLAB*:

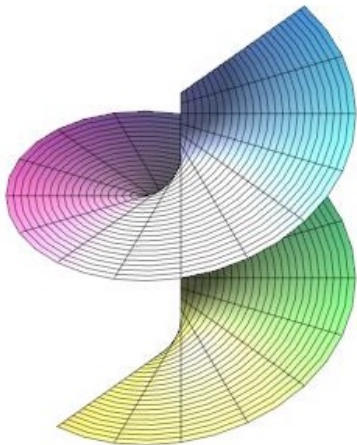
$$\begin{aligned} [u, v] &= \text{meshgrid}(0 : .1 : 1, 0 : .1 : 2 * \pi); \\ \text{surf}(u. * \cos(v), u. * \sin(v), v) \end{aligned}$$

Plotting Parametrized Surface in *Maple*:

$$\text{plot3d}([g1(u, v), g2(u, v), g3(u, v)], u = \dots, v = \dots)$$

Area of a Spiral Ramp

$$g(u, v) = (u \cos v, u \sin v, v), 0 \leq u \leq 1, 0 \leq v \leq 3\pi$$



Area of a Spiral Ramp

$$g(u, v) = (u \cos v, u \sin v, v), 0 \leq u \leq 1, 0 \leq v \leq 3\pi$$

$$g_u = (\cos v, \sin v, 0), g_v = (-u \sin v, u \cos v, 1)$$

$$\begin{aligned} g_u \times g_v &= \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 1 \end{vmatrix} \\ &= \left(\begin{vmatrix} \sin v & 0 \\ u \cos v & 1 \end{vmatrix}, - \begin{vmatrix} \cos v & 0 \\ -u \sin v & 1 \end{vmatrix}, \begin{vmatrix} \cos v & \sin v \\ -u \sin v & u \cos v \end{vmatrix} \right) \\ &= (\sin v, -\cos v, u) \end{aligned}$$

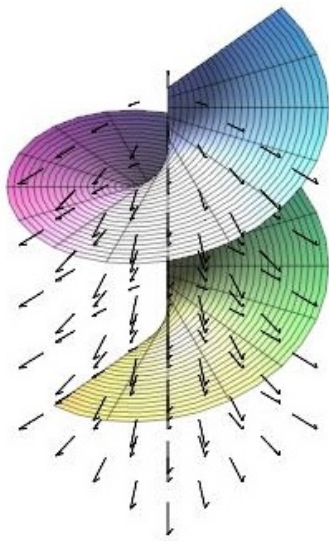
$$\text{Then } |g_u \times g_v| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1 + u^2}$$

$$\text{Area} = \int_{v=0}^{v=3\pi} \int_{u=0}^1 \sqrt{1 + u^2} \, du \, dv$$

If density is $\mu(\mathbf{x}) = u$, then Mass =

$$\begin{aligned} \int_{v=0}^{v=3\pi} \int_{u=0}^1 u(1 + u^2)^{1/2} \, du \, dv &= \int_{v=0}^{v=3\pi} \left[\frac{1}{3}(1 + u^2)^{3/2} \right]_0^1 \, dv \\ &= \int_{v=0}^{v=3\pi} \frac{1}{3} [2^{3/2} - 1^{3/2}] \, dv = 3\pi \frac{1}{3} [2^{3/2} - 1] = \pi [2^{3/2} - 1] \end{aligned}$$

Integrating A Vector Field Over the Spiral Ramp



Integrating A Vector Field Over the Spiral Ramp

$$g(u, v) = (u \cos v, u \sin v, v), 0 \leq u \leq 1, 0 \leq v \leq 3\pi$$

$$g_u = (\cos v, \sin v, 0), g_v = (-u \sin v, u \cos v, 1)$$

$$g_u \times g_v = (\sin v, -\cos v, u)$$

Suppose our vector field is $\mathbf{F}(x, y, z) = (x^2, 0, z^2)$

$$\text{So } F(g(u, v)) = (u^2 \cos^2 v, 0, v^2)$$

The set $D = \{(u, v) : 0 \leq u \leq 1, 0 \leq v \leq 3\pi\}$

We want $\int_D F(g(u, v)) \cdot (g_u \times g_v)$ which equals

$$\begin{aligned} & \int_{v=0}^{3\pi} \int_{u=0}^1 [u^2 \cos^2 v \sin v + uv^2] du dv \\ &= \int_{v=0}^{3\pi} \left[\frac{u^3}{3} \cos^2 v \sin v + \frac{u^2}{2} v^2 \Big|_{u=0}^1 \right] dv = \\ & \int_{v=0}^{3\pi} \left[\frac{1}{3} \cos^2 v \sin v + \frac{1}{2} v^2 \right] dv \\ &= \left[\frac{-\cos^3 v}{9} + \frac{v^3}{6} \right]_{v=0}^{3\pi} = \frac{1}{9} + \frac{3^3 \pi^3}{6} - \frac{-1}{9} = \frac{2}{9} + \frac{9}{2} \pi^3 \end{aligned}$$

Johann Carl Friedrich Gauss



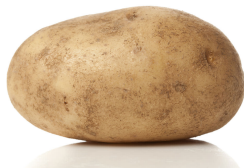
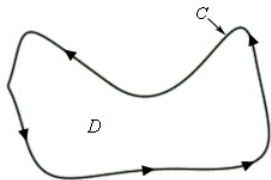
Born: 30 April 30, 1777 in Brunswick, Duchy of Brunswick

Died: 23 February 23, 1855 in Göttingen, Hanover

Biography <http://www.gap-system.org/~history/Biographies/Gauss.html>

Gauss's Theorem aka Divergence Theorem

$$\text{Planar Version: } \int_D \text{div } \mathbf{F} = \int_\gamma \mathbf{F} \cdot \mathbf{N}$$



Three Dimensional Version

∂R is 2-dimensional surface surrounding 3-dimensional region R

$$\int_R \text{div } \mathbf{F} = \int_{\partial R} \mathbf{F} \cdot \mathbf{N}$$

Gauss's Theorem

The Setting

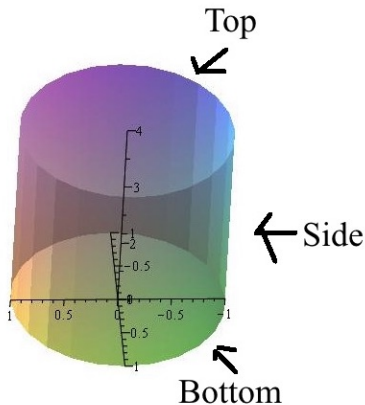
- \mathcal{R} Bounded Solid Region in \mathbb{R}^3
- $\partial\mathcal{R}$ Finitely Many Piecewise Smooth, Closed Orientable Surfaces
Oriented by Unit Normals Pointed away from \mathcal{R}
- \mathbf{F} Continuously Differentiable Vector Field in \mathcal{R}

The Theorem

In this setting
$$\int_{\mathcal{R}} \operatorname{div} \mathbf{F} dV = \int_{\partial\mathcal{R}} \mathbf{F} \cdot d\mathbf{S}$$

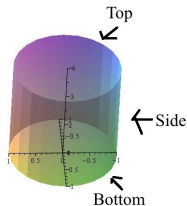
Example Verify Gauss's Theorem where \mathcal{R} is solid cylinder of radius a and height b with the z -axis as the axis of the cylinder and

$$\mathbf{F} = (x, y, z)$$



$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_{Bottom} \mathbf{F} \cdot d\mathbf{S} + \int_{Top} \mathbf{F} \cdot d\mathbf{S} + \int_{Side} \mathbf{F} \cdot d\mathbf{S}$$

Cylinder of Radius a and height b



$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_{Bottom} \mathbf{F} \cdot d\mathbf{S} + \int_{Top} \mathbf{F} \cdot d\mathbf{S} + \int_{Side} \mathbf{F} \cdot d\mathbf{S}$$

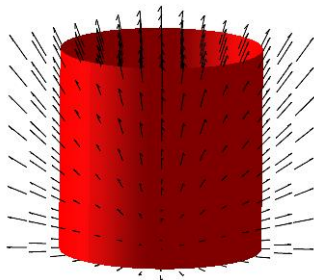
For $\int_{Bottom} \mathbf{F} \cdot d\mathbf{S}$, unit normal is $(0,0,-1)$

Then $(x, y, z) \cdot (0, 0, -1) = -z$ but $z = 0$ so $\int_{Bottom} \mathbf{F} \cdot d\mathbf{S} = 0$

For $\int_{Top} \mathbf{F} \cdot d\mathbf{S}$, unit normal is $(0,0,1)$

Then $(x, y, z) \cdot (0, 0, +1) = z$ but $z = b$ so $\int_{Top} \mathbf{F} \cdot d\mathbf{S}$
is $b \times \text{area of top} = b\pi a^2$

Finally, $\int_{Side} \mathbf{F} \cdot d\mathbf{S}$



Vector Field $\mathbf{F} = (x, y, z)$

Surface: $x^2 + y^2 = a^2, 0 \leq z \leq b$

$g(u, v) = (a \cos u, a \sin u, v), 0 \leq u \leq 2\pi, 0 \leq v \leq b$

Finally, $\int_{Side} \mathbf{F} \cdot dS$

$$g(u, v) = (a \cos u, a \sin u, v), 0 \leq u \leq 2\pi, 0 \leq v \leq b$$

$$g_u = (-a \sin u, a \cos u, 0), \quad g_v = (0, 0, 1)$$

$$g_u \times g_v = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin u & a \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (\text{expanding along bottom row}) (a \cos u, a \sin u, 0)$$

$$\text{Thus } |g_u \times g_v| = \sqrt{a^2 \cos^2 u + a^2 \sin^2 u + 0^2} = a$$

$$\text{Also } F(g(u, v)) = (a \cos u, a \sin u, v) \text{ so } F(g(u, v)) \cdot (g_u \times g_v) = a^2 \cos^2 u + a^2 \sin^2 u + 0 = a^2.$$

$$\text{so } \int_{Side} \mathbf{F} \cdot dS = \int_{v=0}^b \int_{u=0}^{2\pi} a^2 du dv = 2\pi a^2 b$$

Putting it altogether: $\int_S \mathbf{F} \cdot dS$

$$= \int_{Bottom} \mathbf{F} \cdot dS + \int_{Top} \mathbf{F} \cdot dS + \int_{Side} \mathbf{F} \cdot dS = 0 + \pi a^2 b + 2\pi a^2 b = 3\pi a^2 b$$

On The Other Hand, we compute $\int_R \operatorname{div} \mathbf{F}$

$$\mathbf{F} = (x, y, z)$$

$$\operatorname{div} \mathbf{F} = 1 + 1 + 1 = 3$$

The solid R is more easily described in polar coordinates

$$0 \leq \theta \leq 2\pi \quad 0 \leq r \leq a \quad 0 \leq z \leq b.$$

$$\int_R \operatorname{div} \mathbf{F} = \int_{\theta=0}^{2\pi} \int_{z=0}^b \int_{r=0}^a \operatorname{div} \mathbf{F} r dr dz d\theta = \int_{\theta=0}^{2\pi} \int_{z=0}^b \int_{r=0}^a 3r dr dz d\theta$$

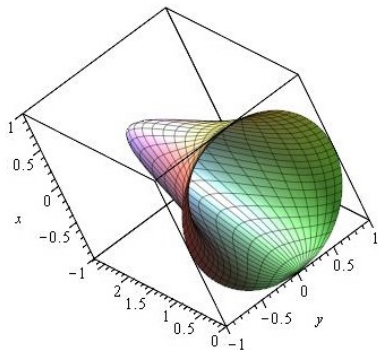
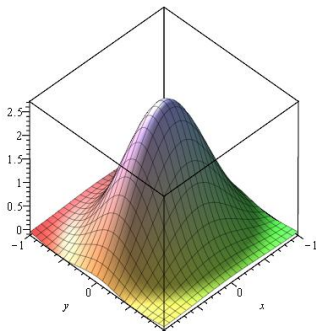
$$\begin{aligned} \int_{\theta=0}^{2\pi} \int_{z=0}^b 3 \frac{r^2}{2} \Big|_{r=0}^a dz d\theta &= \int_{\theta=0}^{2\pi} \int_{z=0}^b \frac{3}{2} a^2 dz d\theta = \int_{\theta=0}^{2\pi} \frac{3}{2} a^2 b d\theta = 2\pi \frac{3}{2} a^2 b \\ &= 3a^2 b \pi \end{aligned}$$

Example: $\mathbf{F} = (e^y \cos z, \sqrt{x^3 + 1} \sin z, x^2 + y^2 + 3)$

$$\operatorname{div} \mathbf{F} = 0 + 0 + 0 = 0$$

so $\int_R \operatorname{div} \mathbf{F} = 0$ for any region in \mathbb{R}^3 .

Let S be graph of $z = (1 - x^2 - y^2)e^{1-x^2-3y^2}$ for $z \geq 0$
oriented by outward pointing unit normal vector.



Finding $\int_S \mathbf{F} \cdot d\sigma$ directly is impossible.

A Clever Way To Find $\int_S \mathbf{F} \cdot d\sigma$ indirectly.

Cap the Surface with a Disk so New Surface Bounds a
3-Dimensional Region

Form closed surface $S \cup S'$ where S' is the disk of radius 1
($x^2 + y^2 = 1$) in $z = 0$ plane. Then $\int_{\partial R} \mathbf{F} = \int_{S \cup S'} \mathbf{F} = \int_S \mathbf{F} + \int_{S'} \mathbf{F}$

But by Gauss's Theorem, this integral equals 0.

$$\text{Hence } \int_S \mathbf{F} = - \int_{S'} \mathbf{F}$$

Now

$$\begin{aligned} \int_{S'} \mathbf{F} &= - \int (-, -, x^2 + y^2 + 3) \cdot (0, 0, -1) = \int x^2 + y^2 + 3 \, dx \, dy \\ &= \int_{\theta=0}^{2\pi} \int_{r=0}^1 (r^2 + 3) r \, dr \, d\theta = \frac{7}{2}\pi \end{aligned}$$

Next Time:

Stokes's Theorem

$$\int_S \operatorname{curl} \mathbf{F} = \int_{\partial S} \mathbf{F}$$

S is a Surface in \mathbb{R}^3

Theorem: A continuously differentiable gradient field has a symmetric Jacobian matrix.

Proof: If \mathbf{F} is a gradient field, then $\mathbf{F} = \nabla f$ for some real-valued function f . Then $\mathbf{F} = (f_x, f_y)$ so the Jacobian matrix is

$$J = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

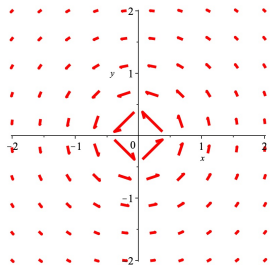
By Continuity of Mixed Partial, $f_{xy} = f_{yx}$ so J is symmetric. \square

Theorem: If \mathbf{F} is conservative, then its Jacobian is symmetric.

Theorem: If \mathbf{F} is conservative, then its Jacobian is symmetric.

The converse (Symmetric Jacobian Implies Conservative) is
FALSE in general.

Example: Consider the vector field $\mathbf{F}(x, y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$



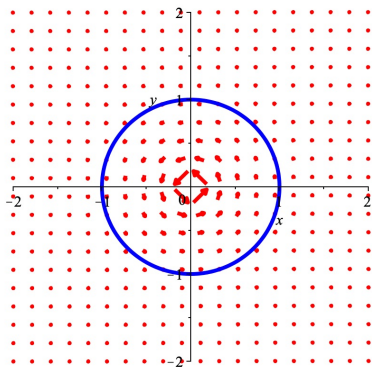
defined for all $(x, y) \neq (0, 0)$

$$\text{Then Jacobian} = \begin{pmatrix} - & \frac{y^2 - x^2}{(x^2 + y^2)^2} \\ \frac{y^2 - x^2}{(x^2 + y^2)^2} & - \end{pmatrix}$$

$$\mathbf{F}(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$$

Has a Symmetric Jacobian But Is Not Conservative!

If \mathbf{F} were conservative, then the line integral of \mathbf{F} around any closed loop would be 0. Consider γ the unit circle as a loop running counterclockwise starting and ending at $(1,0)$.



$$\mathbf{F}(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$$

γ : unit circle as a loop running counterclockwise starting and ending at (1,0).

We parametrize γ by $g(t) = (\cos t, \sin t)$, $0 \leq t < 2\pi$ so that $g'(t) = (-\sin t, \cos t)$ and

$$\mathbf{F}(g(t)) = \left(\frac{-\sin t}{\cos^2 t + \sin^2 t}, \frac{\cos t}{\cos^2 t + \sin^2 t} \right) = (-\sin t, \cos t)$$

$$\mathbf{F}(g(t)) \cdot g'(t) = (-\sin t, \cos t) \cdot (-\sin t, \cos t) = \sin^2 t + \cos^2 t = 1$$

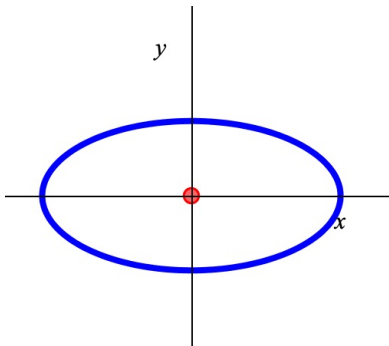
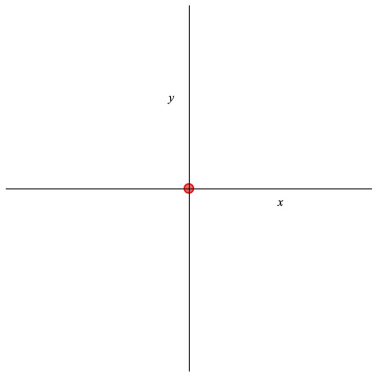
$$\text{Thus } \int_{\gamma} \mathbf{F} = \int_0^{2\pi} 1 \, dt = 2\pi \neq 0.$$

What is Wrong the Vector Field

$$\mathbf{F}(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)?$$

The Domain of the Vector Field

(Plane minus the Origin)
Is Not Simply Connected.

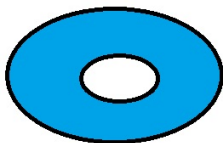


Simple Connectedness

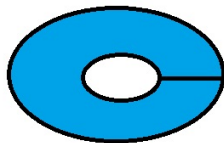
A set B is **simply connected** if every closed curve in B can be continuously contracted to a point in such a way as to stay in B during the contraction. More precisely,

Definition: An open set B is **simply connected** if every piecewise smooth closed curve lying in B is the border of some piecewise smooth orientable surface S lying in B , and with parameter domain a disk in \mathcal{R}^2 .

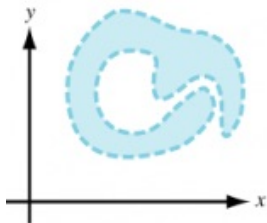
Theorem: Let \mathbf{F} be a continuously differentiable vector field defined on an open set B in \mathcal{R}^2 or \mathcal{R}^3 . If B is simply connected and $\text{curl } \mathbf{F}$ is identically zero in B , then \mathbf{F} is a gradient field in B ; that is, there is a real-valued function f such that $\mathbf{F} = \nabla f$



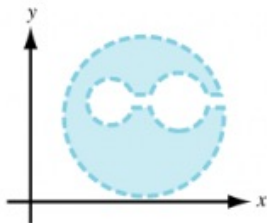
not simply connected



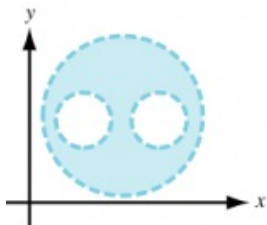
simply connected
thanks to single cut



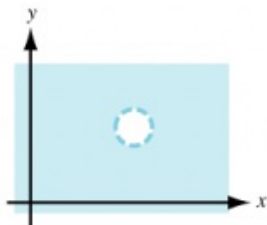
(a) A simply connected domain



(b) A simply connected domain

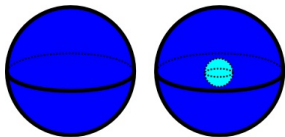


(c) A multiply connected domain

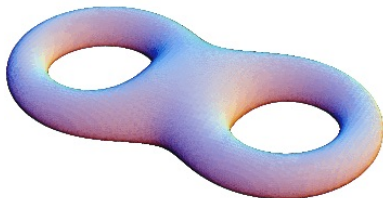
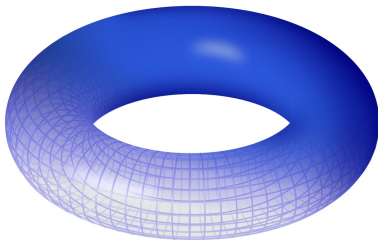
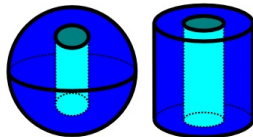


(d) A multiply connected domain

Simply connected



Non-simply connected





Theorem: Let \mathbf{F} be a continuously differentiable vector field defined on an open set B in \mathcal{R}^2 or \mathcal{R}^3 . If B is simply connected and $\text{curl } \mathbf{F}$ is identically zero in B , then \mathbf{F} is a gradient field in B ; that is, there is a real-valued function f such that $\mathbf{F} = \nabla f$