MATH 224

Some Hints and Answers for Assignment 21

Exercises 52 and 53 of Chapter 5; Exercises 1ac, 2 ac, 3, and 4d of Chapter 6.

52: Replace x with $r\cos\theta$ and y with $r\sin\theta$, noting that $x^2+y^2=r^2$
(a) $5\cos\theta = 2r(4+3r\cos\theta\, r\sin\theta$

(b) Simplify to get

$$r^2 = \frac{12+2\tan\theta}{\sin^2\theta = 7\cos^2\theta}$$

53: (a) Vertical line through (8,0)
(b) Horizontal line through (0, -4)
(c) Circle with center $(a/2, 0$ and radius $a/2$.
(d) Circle with center $(0, b/2)$ and radius $(0, b/2)$.

Exercises from Chapter 6

1: (a) If x is bounded above and below by 0 and 3 and y by $2+x$ and $2-x$ then the area being integrated over is a triangle with vertices $(0,2)$, $(3,-1)$, and $(3,5)$. If we integrate with respect to y first to create vertical strips, each strip runs from the line $y=2-x$ to $y=2+x$.
If we want to integrate with respect to x first to create horizontal strips there will be two different types of strips. When y is between -1 and 2, the horizontal strips are bounded by the lines $x=y-2$ and $x=3$. When y is between 2 and 5 the horizontal strips are bounded by $x=y-2$ and $x=3$.
(c) The region described by the inequalities is the area between the lines $y=x$ and $y=x^2$ where x is in the interval $[0,1]$. Notice that this set is both x-simple and y-simple, so integrating in either order will require only one iterated integral. If we wish to integrate with respect to y first, the vertical strips will be bounded below by $y=x^2$ and above by $y=x$.
If we wish to integrate with respect to x first we must rewrite the bounds for y so that they are bounds for x in terms of y. If $y\le x$ then $x\ge y$. If $y\ge x^2$ then $x\le\sqrt{y}$. The bounds for y will now be the same as the bounds for x in the first integral: $0\le y\le 1$.

2: Sketch the region of integration described by each of the iterated integrals below and write the appropriate iterated integral with the order of integration reversed: (a) $\int_1^3\int_0^{6-2x} f(x,y)\,dy\,dx$ and (c) $\int_0^2\int_{1+y/2}^2 f(x,y)\,dx\,dy$

Solution: 2. a) The first integral here, the inner integral, integrates $f(x,y)$ as y runs from 0 to $6-2x$. The area of integration then has one set of boundaries given by $0\le y\le 6-2x$. The outer integral integrates $\int f(x,y)\,dy$ as x runs from 1 to 3. The second set of constraints of our area of integration is then $1\le x\le 3$.

If we would like to exchange the order of integration, we must find boundaries for x in terms of y and then constant boundaries for y. Recall from the chapter that integrating with respect to x first will leave a function g of y which has $g(y)$ equal $f(x,y)$ integrated over a horizontal line running across the area of integration at a particular value of y. If we draw horizontal lines across our area we see that, for all values of y, the lower bound is 1 and the upper bound is $3-\frac{y}{2}$. The inner integral we are looking for is then

$$\int_{x=1}^{x=3-\frac{y}{2}} f(x,y)\,dx.$$

Of all points (x,y) in the set $1\le x\le 3-\frac{y}{2}$, the only points we would like to integrate over are those for which $0\le y\le 4$. The entire iterated integral is then

$$\int_{y=0}^{y=4}\int_{x=1}^{x=3-\frac{y}{2}} f(x,y)\,dx\,dy.$$

c) The inner integral here has bounds $1+\frac{y}{2}\le x\le 2$. Rewriting this first inequality to find y in terms of x we have $y\le 2x-2$. If y is bound by 0 and 2 then the entire area of integration is the triangle bound by the lines $y=2x-2$, $y=0$, and $x=2$.

The given iterated integral integrates first with respect to x and then with respect to y. That is, the given integral uses the horizontal line method. If we use the vertical line method we can see that each vertical line for a given value of x has a lower bound of 0 and an upper bound of $2x-2$. Furthermore, the vertical lines run from $x = 1$ to $x = 2$. The iterated integral in reverse order is then

$$\int_{x=1}^{x=2} \int_{y=0}^{y=2x-2} f(x,y)\, dy\, dx$$

3: (a) $\int_0^4 \int_0^y y^2\, dx\, dy = 64$

(b): $\int_{-4}^{4} \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} 5\, dx\, dy = 80\pi$. Sketch the area of integration. Note that at some point you should recognize of a circle of radius 4 whose area you know.

4ad: a) Answer is 9/2. Begin by sketching the region bounded by the two curves.

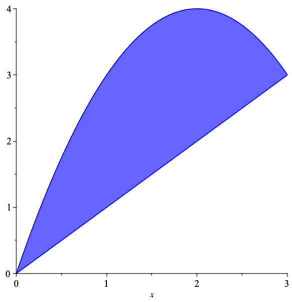

Figure 1: The area bounded by $y = x$ and $y = 4x - x^2$.

This region is both x-simple and y-simple so either order of integration should be straightforward.

d)

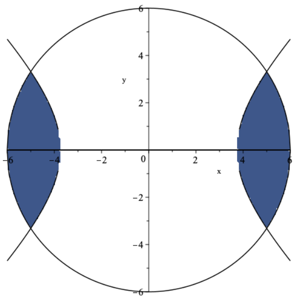

Figure 2: The area bounded by $x^2 - y^2 = 14$ and $x^2 + y^2 = 36$.

Notice here that the area we would like to find is actually two separate areas of equivalent size. Furthermore, both of these smaller regions are split in half by the x axis. This allows us to solve for the total area by solving for one quarter and multiplying by 4. Solve for the area of the quarter which lies completely in the first quadrant. Because we are dealing with an area described by a circular boundary, converting to polar coordinates and applying Jacobi's Theorem will give us simpler expressions to integrate. In polar coordinates, the circular curve is described by $r = 6$ and the hyperbolic curve is

$$r^2\cos^2\theta - r^2\sin^2\theta = 14 \Rightarrow r^2(\cos^2\theta - \sin^2\theta) = 14$$

Applying the double angle formula to the right hand side we have

$$r^2\cos^2\theta - r^2\sin^2\theta = 14 \Rightarrow r = \sqrt{\frac{14}{\cos 2\theta}}.$$

The total area is 19.8.