MATH 224

Some Hints and Answers for Assignment 22

Exercises 8, 9, 10, 11 and 12 in Chapter 6

8: Determine the integral of $f(x, y) = x^2 + y^2$ over the region bounded by the x-axis and the top half of the unit circle centered at the origin.
Solution: Figure 1 shows the region. Carving thee region into vertical lines, we see that or each x between -1 and 1, a vertical segment runs from the horizontal axis up to the semicircle; that is, $y = 0$ to $y = \sqrt{1 - x^2}$. You may get an integral that can be solved in a variety of ways including integration by parts, the substitution $x = \sin\theta$, and the recognition that $\int_{-1}^{1} \sqrt{1 - x^2}\, dx$ is the area $\pi/2$ of a semicircle of radius 1. Final answer is $\pi/4$.

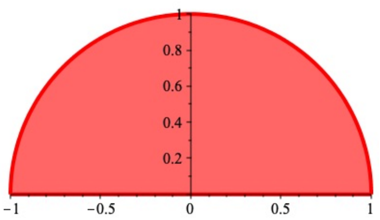


Figure 1: Region of Exercise 8

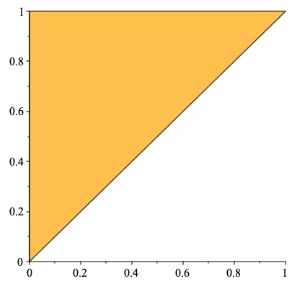


Figure 2: Region of Exercise 9

9: Find the value of the integral of $f(x, y) = x^2 + y^2$ over the region enclosed by the triangle with vertices (0,0), (0,1), and (1,1).
Solution: Figure 2 shows the region. Each horizontal slice runs from $x = 0$ to $x = y$ and we have a horizontal slice for each y from 0 to 1. Final answer is $\frac{1}{3}$

10: Evaluate the integral of $2x + 3y + 4z$ over the region enclosed by the tetrahedron with vertices (0,0,0), (0,0,3), (0,2,0), and (1,0,0).

Solution: Figure 3 shows the tetrahedron. Three of its four sides are the coordinate planes and the fourth is the plane with equation $\frac{x}{1} + \frac{y}{2} + \frac{z}{3} = 1$.. You can set up the order of integration in 6 possible ways. We'll do it as $\iiint 2x + 3y + 4z\, dz\, dy\, dx$. Figures 4, 5 and 6 display the xy, xz and yz slices respectively.

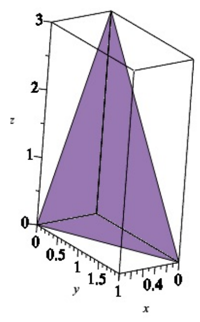
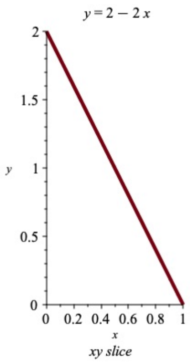
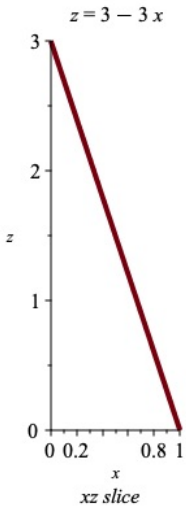


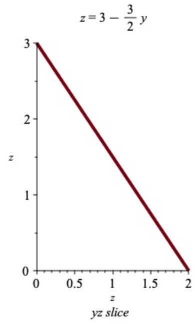
Figure 3



xy- slice; $z = 0$
$y = 2 - 2x$
Figure 4



xz-slice: $y = 0$
$z = 3 - 3x$
Figure 5



yz-slice:x $= 0$
$z = 3 - \frac{3}{2}y$
Figure 6

The value of the triple integral is 5.

11: Determine the volume bounded by the coordinate axes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.
Solution: Proceed as in Exercise 10. Vertices of the region are $(a, 0, 0), (0, b, 0), (0, 0, c)$. Volume is $\frac{abc}{6}$

12: Find the volume of the solid bounded by the surfaces $y^2 + z^2 = 4ax, x = 3a$, and $y^2 = ax$.

The equations $x = 3a$ and $y^2 = ax$ define a figure in the plane bounded by a parabola and a straight line segment. See Figure 7 in red below. The line segment and the parabola intersect at the points $(3a, \pm\sqrt{3}a)$. A double integral over this region would be written as

$$\int_{x=0}^{x=3a} \int_{y=-\sqrt{ax}}^{\sqrt{ax}} f(x, y)\, dy\, dx$$

if we imagine the region carved into vertical slices.

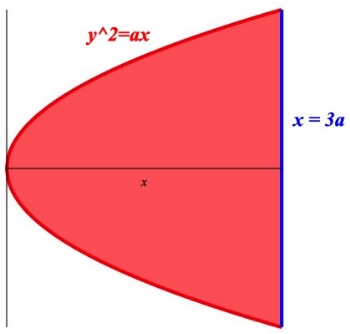


Figure 7

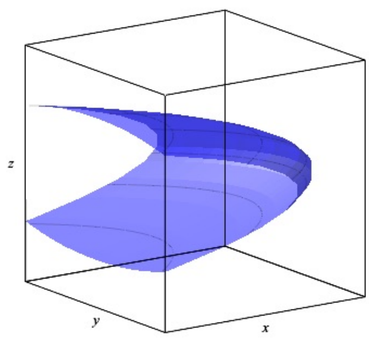


Figure 8

Figure 8 displays a graph of the surface defined by $y^2 + z^2 = 4ax$.
We can solve the remaining equation $y^2 + z^2 = 4ax$ for z in terms of x and y: $z = \pm\sqrt{4ax - y^2}$. Volume $= a^3 \left[3\pi + \frac{9}{2}\sqrt{3}\right]$