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MATH 0224: Vector Calculus
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Location coordinates:

Middlebury $(44.0153^\circ, \ -73.1673^\circ)$
Dallas $(32.7792^\circ, \ -96.8089^\circ)$
Tampa $(27.9642^\circ, \ -82.4526^\circ)$
Detroit $(42.3314^\circ, \ -83.0458^\circ)$

Set up a coordinate system:
$x = 50\ (longitude) \quad y = 70\ (latitude)$

Middlebury:
$x_M = 50 \times -73.1673 = -3658.365$
$y_M = 70 \times 44.0153 = 3081.071$

$M = (-3658.365, 3081.071)$

Dallas:
$x_D = 50 \times -96.8089 = -4840.445$
$y_D = 70 \times 32.7792 = 2294.544$

$D = (-4840.445, 2294.544)$

Tampa:
$x_T = 50 \times -82.4526 = -4122.630$
$y_T = 70 \times 27.9642 = 1957.494$

$T = (-4122.630, 1957.494)$

Detroit:
$x_{De} = 50 \times -83.0458 = -4152.290$
$y_{De} = 70 \times 42.3314 = 2963.198$

$De = (-4152.290, 2963.198)$

Let an arbitrary position be the location of the base office

$$P = (x, y)$$

For each city, $P_i = (a_i, b_i)$ the vector from the city to the base is given by

$$P - P_i = (x - a_i, y - b_i)$$

An arbitrary point with respect to each location:

Middlebury:

$P_M(x, y) = (x + 3658.365,\ y - 3081.071)$

$d_M(x, y) = \sqrt{(x + 3658.365)^2 + (y - 3081.071)^2}$

Dallas:

$P_D(x, y) = (x + 4840.445,\ y - 2294.544)$

$d_D(x, y) = \sqrt{(x + 4840.445)^2 + (y - 2294.544)^2}$

Tampa:

$P_T(x, y) = (x + 4122.630,\ y - 1957.494)$

$d_T(x, y) = \sqrt{(x + 4122.630)^2 + (y - 1957.494)^2}$

Detroit:

$P_{De}(x, y) = (x + 4152.290,\ y - 2963.198)$

$d_{De}(x, y) = \sqrt{(x + 4152.290)^2 + (y - 2963.198)^2}$

Each trip is a round trip from home base to the city and back, so every trip contributes: $2 \times (distance)$

Trips to each location:
3 trips to Middlebury
2 trips to Dallas
4 trips to Tampa
1 trip to Detroit

Total monthly mileage:

$$L(x, y) = 2(3d_M(x, y) + 2d_D(x, y) + 4d_T(x, y) + 1d_{De}(x, y))$$

6.74$ per mile, therefore, the monthly cost function is:

$$C(x, y) = 6.74L(x, y)$$

Therefore:

$$C(x, y) = 13.48(3d_M(x, y) + 2d_D(x, y) + 4d_T(x, y) + 1d_{De}(x, y))$$

To **find the critical point**, we will minimize the function

$$f(x, y) = 3d_M + 2d_D + 4d_T + 1d_{De} \text{ such that } \nabla f(x, y) = 0$$

Since the constant 13.48 does not change where the minimum occurs, we omit that from the equation.

The distance between an arbitrary point P_i and each location is described as

$$d_i = \sqrt{(x - a_i)^2 + (y - b_i)^2}$$

Where $P_i = (a_i, b_i)$ are the coordinates of each location.

When differentiating, d_i we get

$$\frac{\delta d_i}{\delta x} = \frac{x - a_i}{\sqrt{(x-a_i)^2 + (y-b_i)^2}} = \frac{x-a_i}{d_i}$$

$$\frac{\delta d_i}{\delta y} = \frac{y - b_i}{\sqrt{(x-a_i)^2 + (y-b_i)^2}} = \frac{y-b_i}{d_i}$$

$$\nabla d_i = \left(\frac{x-a_i}{d_i}, \frac{y-b_i}{d_i}\right) = \left(\frac{x-a_i, y-b_i}{d_i}\right)$$

The function f is in the general form of

$$f(x, y) = \sum_i w_i d_i$$

where w_i is the number of times traveled to a city, and d_i is the distance to that city.

As we can see, the numerator ∇d_i of is the vector from the base location to a city, and the denominator is the distance between the two.

Therefore, the gradient of f is

$$\nabla f(x, y) = \sum_i w_i \frac{P-P_i}{||P-P_i||} = 0$$

In order to **minimize this function**, the gradient should be equal to 0.

$$\nabla f(x, y) = 3\frac{P-P_M}{||P-P_M||} + 2\frac{P-P_D}{||P-P_D||} + 4\frac{P-P_T}{||P-P_T||} + \frac{P-P_{De}}{||P-P_{De}||} = 0$$

For x: $3\frac{x+3658.365}{d_M} + 2\frac{x+4840.445}{d_D} + 4\frac{x+4122.630}{d_T} + \frac{x+4152.290}{d_{De}} = 0$

For y: $3\frac{y-3081.071}{d_M} + 2\frac{y-2294.544}{d_M} + 4\frac{y-1957.494}{d_M} + \frac{y-2963.198}{d_M} = 0$

Therefore, the minimum position is $P^*(x, y) = (-4152.283,\ 2178.447)$

Convert the position: $longitude = \frac{x}{50} = \frac{-4152.283}{50} = -83.0457°$

$latitude = \frac{y}{70} = \frac{2178.447}{70} = 31.1207°$

Therefore, the minimum position is $(31.1207°, -83.0457°)$, which is $(31.1207°N, 83.0457°W)$.

This point lies near **Lakeland, Georgia**, and represents the location that minimizes the total monthly cost.

This result is consistent with the model, as the optimal point is located closest to Tampa, which has the highest travel frequency (4 trips per month), thereby exerting the greatest influence on the optimal location.

To show that this point is a minimum and not a maximum or saddle point, we will use the Hessian matrix.

$$C(x, y) = 13.48(3d_M(x, y) + 2d_D(x, y) + 4d_T(x, y) + 1d_{De}(x, y))$$

which can be rewritten in the form $C(x, y) = 13.48\sum_i w_i d_i$

Find the second derivative of the d_i component of the function $C(x, y)$

Find $d_{i,xx}$:

$\frac{\delta d_i}{\delta x} = \frac{x-a_i}{d_i}$

$$\frac{\delta^2 d_i}{\delta x^2} = \frac{d_i - (x-a_i)\frac{\delta d_i}{\delta x}}{d_i^2}$$

$$d_{i,xx} = \frac{d_i - (x-a_i)\frac{x-a_i}{d_i}}{d_i^2} = \frac{d_i - \frac{(x-a_i)^2}{d_i}}{d_i^2} = \frac{(x-a_i)^2 + (y-b_i)^2 - (x-a_i)^2}{d_i^3} = \frac{(y-b_i)^2}{d_i^3}$$

Find $d_{i,yy}$:

Using the same steps used for finding $d_{i,xx}$, we'll get

$$d_{i,yy} = \frac{(x-a_i)^2}{d_i^3}$$

Find $d_{i,xy}$:

$$d_{i,xy} = (x - a_i)\frac{\delta}{\delta y}\left(\frac{1}{d_i}\right)$$

$$\frac{\delta}{\delta y}\left(\frac{1}{d_i}\right) = -\frac{1}{d_i^2}\frac{\delta d_i}{\delta y} = -\frac{1}{d_i^2}\frac{y-b_i}{d_i} = -\frac{y-b_i}{d_i^3}$$

$$d_{i,xy} = -\frac{(x-a_i)(y-b_i)}{d_i^3}$$

Using the second derivatives of d_i, we can find the Hessian entries using $C(x, y)$:

$$H_C(x,y) = \begin{pmatrix} C_{xx} & C_{xy} \\ C_{yx} & C_{yy} \end{pmatrix}$$

$$C_{xx} = 13.48\sum_i w_i \frac{(y-b_i)^2}{d_i^3}$$

$$C_{yy} = 13.48\sum_i w_i \frac{(x-a_i)^2}{d_i^3}$$

$$C_{xy} = 13.48\sum_i w_i \left[-\frac{(x-a_i)(y-b_i)}{d_i^3}\right]$$

Plug in the minimizing point $P^*(x, y) = (-4152.283,\ 2178.447)$

Calculate Hessian for each individual location:

$$H_M \approx \begin{pmatrix} 0.03025 & -0.01655 \\ -0.01655 & 0.00906 \end{pmatrix}$$

$$H_D \approx \begin{pmatrix} 0.00107 & 0.00634 \\ 0.00634 & 0.03756 \end{pmatrix}$$

$$H_T \approx \begin{pmatrix} 0.23759 & 0.03189 \\ 0.03189 & 0.00428 \end{pmatrix}$$

$$H_{De} \approx \begin{pmatrix} 0.01718 & 0.00000 \\ 0.00000 & 0.00000 \end{pmatrix}$$

Adding weighted contributions:

$$H_C(P^*) = H_M + H_D + H_T + H_{De}$$

$$H_C(P^*) \approx \begin{pmatrix} 0.28608 & 0.02167 \\ 0.02167 & 0.05090 \end{pmatrix}$$

The matrix is *positive definite* if:

$$A > 0$$

$$AD - B^2 > 0$$

Checking for our points:

$$0.28608 > 0$$

$$0.28608 \times 0.05090 - 0.02167^2 = 0.01409 > 0$$

The matrix is positive definite, therefore the point is a minimum.

Minimum monthly cost:

Distances in relation to the point:

$P^*(x, y) = (-4152.283,\ 2178.447)$

$$d_M(P^*) = \sqrt{(-4152.283 + 3658.365)^2 + (2178.447 - 3081.071)^2} = 1028.924$$

$$d_D(P^*) = \sqrt{(-4152.283 + 4840.445)^2 + (2178.447 - 2294.544)^2} = 697.886$$

$$d_T(P^*) = \sqrt{(-4152.283 + 4122.630)^2 + (2178.447 - 1957.494)^2} = 222.935$$

$$d_{De}(P^*) = \sqrt{(-4152.283 + 4152.290)^2 + (2178.447 - 2963.198)^2} = 784.751$$

Total monthly mileage

$$L(x,y) = 2(3d_M(P^*) + 2d_D(P^*) + 4d_T(P^*) + 1d_{De}(P^*))$$

$$L(x,y) = 2(3(1028.924) + 2(697.886) + 4(222.935) + 1(784.751)) = 12318.07 \text{ mi}$$

$$C(x,y) = 6.74L(x,y) = 6.74 \times 12318.07 = \$83023.792$$

The cost for individual locations:

If the home base were in Middlebury:

$$d_M(x,y) = \sqrt{(x + 3658.365)^2 + (y - 3081.071)^2}$$

$$d_{M-D}(x,y) = \sqrt{(-4840.445 + 3658.365)^2 + (2294.544 - 3081.071)^2} = 1419.837$$

$$d_{M-T}(x,y) = \sqrt{(-4122.630 + 3658.365)^2 + (1957.494 - 3081.071)^2} = 1215.716$$

$$d_{M-De}(x,y) = \sqrt{(-4152.290 + 3658.365)^2 + (2963.198 - 3081.071)^2} = 507.795$$

$$C_M = 13.48(3 \times 0 + 2 \times 1419.837 + 4 \times 1215.716 + 507.795) = \$110675.289$$

Compared to Middlebury: $\$110675.289 - \$83023.792 = \$27651.497$ saved per month

If the home base were in Dallas:

$$d_D(x,y) = \sqrt{(x + 4840.445)^2 + (y - 2294.544)^2}$$

$$d_{D-M}(x,y) = 1419.837$$

$$d_{D-T}(x,y) = \sqrt{(-4122.630 + 4840.445)^2 + (1957.494 - 2294.544)^2} = 793.008$$

$$d_{D-De}(x,y) = \sqrt{(-4152.290 + 4840.445)^2 + (2963.198 - 2294.544)^2} = 959.508$$

$$C_D = 13.48(3 \times 1419.837 + 2 \times 0 + 4 \times 793.008 + 959.508) = \$113111.368$$

Compared to Dallas: $\$113111.368 - \$83023.792 = \$30087.576$ saved per month

If the home base were in Tampa:

$$d_T(x,y) = \sqrt{(x + 4122.630)^2 + (y - 1957.494)^2}$$

$$d_{T-M}(x,y) = 1215.717$$

$$d_{T-D}(x,y) = 793.008$$

$$d_{T-De}(x,y) = \sqrt{(-4152.290 + 4122.630)^2 + (2963.198 - 1957.494)^2} = 1006.141$$

$$C_T = 13.48(3 \times 1215.717 + 2 \times 793.008 + 4 \times 0 + 1006.141) = \$84105.872$$

Compared to Tampa: $\$84105.872 - \$83023.792 = \$1082.080$ saved per month

If the home base were in Detroit:

$d_{De}(x, y) = \sqrt{(x + 4152.290)^2 + (y - 2963.198)^2}$

$d_{De-M}(x, y) = 507.795$

$d_{De-D}(x, y) = 959.508$

$d_{De-T}(x, y) = 1006.141$

$$C_{De} = 13.48(3 \times 507.795 + 2 \times 959.508 + 4 \times 1006.141) = \$100654.688$$

Compared to Detroit: $\$100654.688 - \$83023.792 = \$17630.896$ saved per month

After comparing the four cities as potential home base locations, the optimal location near Lakeland, Georgia provides the lowest overall cost. Among the four cities, Tampa yields the lowest cost, but it is still more expensive than the optimal location.